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The perturbative $SO(3)$ invariant of rational homology 3-spheres recovers from the universal perturbative invariant☆

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Abstract

For a Lie algebra $\mathbf{g}$ and its representation R, the quantum $(\mathbf{g}, R)$ invariant of knots recovers from the Kontsevich invariant through the weight system derived from substitution of $\mathbf{g}$ and R into chord diagrams. We expect a similar property for invariants of 3-manifolds; for a Lie group G, the perturbative G invariant of 3-manifolds should recover from the universal perturbative invariant defined in [25] through the weight system derived from substitution of the Lie algebra of G. In this paper we give a rigorous proof of the recovery for $G = SO(3)$.

Witten's path integral formula [39] gave the "definition" of the quantum G invariant of 3-manifolds for each Lie group G. Though the path integral is not mathematically justified yet, the formula let us be confident of the existence of rigorous constructions of the quantum invariants of 3-manifolds. In fact, after Witten's formula, many researchers gave rigorous construction of the quantum invariants of 3-manifolds [3, 9–12, 26, 32, 37, 38, 40].

Since we have many Lie groups, we had obtained many invariants of 3-manifolds. Further these invariants might behave independently, i.e., there might not exist a linear order with respect to strength of the invariants. So we consider the following two approaches to control the invariants.

- Unify the invariants into an invariant.
- Characterize the invariants with a property.

As for the quantum invariants of knots, we obtained the Kontsevich invariant and Vassiliev invariants by the two approaches, i.e., the quantum invariants of knots were unified into the

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Kontsevich invariant, see Theorem 1.2, and were characterized by Vassiliev invariants such that the coefficients of quantum invariants are Vassiliev invariants, when expanded into power series of a free parameter h.

Unfortunately the quantum G invariant, denoted by $\tau_r^G(M)$, belongs to $\mathbb{C}$, which is not a polynomial ring nor a power series ring, unlike the case of knots. Since we can not expand $\tau_r^G(M)$ with respect to r in an ordinary sense, we have a technical difficulty to consider invariants like Vassiliev invariants for these quantum invariants. Instead of expansion in r, we consider the perturbative invariant $\tau^G(M) \in \mathbb{Q}[[h]]$ obtained from $\tau_r^G(M)$ by taking r-adic limit; for rigorous definition of the perturbative invariant for $G = SO(3)$ and rational homology 3-spheres M, see [29]. See also [33–35] for the perturbative invariants.

In a similar way as we consider the Kontsevich invariant and Vassiliev invariants for quantum invariants of knots, we consider the universal perturbative invariant and finite type invariants for the perturbative invariants of 3-manifolds. The universal perturbative invariant was defined in [25], and finite type invariants were defined in [30] for integral homology 3-spheres. Similarly as in the case of knots, we expect two universalities of the universal perturbative invariant, see Fig. 1. The first universality is the one among the perturbative invariants, conjectured in [25]. The second one is the one among finite type invariants, which was shown by Thang Le [16].

In this paper our aim is to show the recovery of the perturbative $SO(3)$ invariant from the universal perturbative invariant (Theorem 2.7). This is the positive answer to the conjecture on the first universality for $G = SO(3)$. As a corollary, we show a relation between the coefficients of the perturbative $SO(3)$ invariant and finite type invariants (Theorem 3.3).

In Section 1 we review the recovery of the quantum invariants of links from the Kontsevich invariant, and show properties which are necessary in the proof of the main theorem. In section 2 we prove the main theorem of the recovery of the perturbative $SO(3)$ invariant from the universal perturbative invariant. As a corollary of the main theorem, we show that the coefficients of the perturbative $SO(3)$ invariant are finite type in Section 3.

1. Recovery of the quantum sl_2 invariant of links

In this section we review properties of the Kontsevich invariant of links and show a direct proof of the recovery of the quantum sl_2 invariant of links from the Kontsevich invariant.

1.1. The modified Kontsevich invariant of links

For a compact 1-manifold X, a *chord diagram* on X is a uni-trivalent graph such that its univalent vertices are on X and its trivalent vertices are vertex-oriented, where we call that a trivalent vertex is vertex-oriented if a cyclic order of the three edges around the vertex is fixed. We regard homeomorphic chord diagrams as the same one. In pictures, we depict X with solid lines, and depict the uni-trivalent graph with dashed lines. Further, $\mathscr{A}(X)$ denotes the vector space over $\mathbb{C}$ spanned by chord diagrams on X subject to the AS, IHX and STU relations. We have operations $\Delta_{(C)}$ and $\varepsilon_{(C)}$ on the space of chord diagrams on X, fixing a component C of X. For detailed definitions of these notations, see [25].

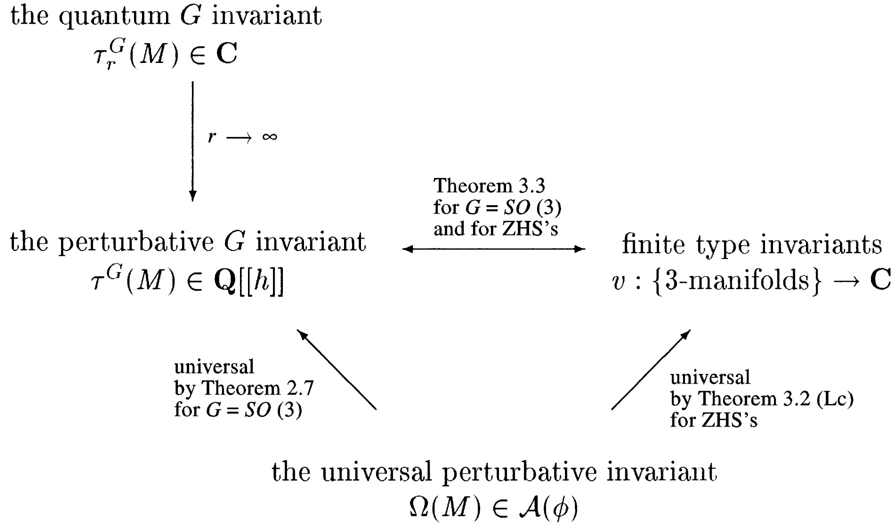


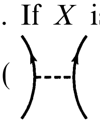
Fig. 1. Invariants of 3-manifolds and the relations between them.

For an oriented framed l-component link in S^3, $\hat{Z}(L) \in \mathscr{A}(\sqcup^l S^1)$ denotes the modified Kontsevich invariant; see [18–21] for its combinatorial definition and its properties. In particular, we have

Proposition 1.1 (Le and Murakami [21]). *Let C be a component of an oriented framed l-component link L, and C' the corresponding component of $\sqcup^l S^1$.*

1. *The formula $\hat{Z}(L') = \Delta_{(C')}(\hat{Z}(L))$ holds, where L' is the link obtained from L by taking 2-parallel along C, i.e., by replacing C with two copies of C which are located in the parallel position along the framing of C.*
2. *The formula $\hat{Z}(L - C) = \varepsilon_{(C')}(\hat{Z}(L))$ holds, where $L - C$ denotes the link obtained from L by removing the component C.*

1.2. Recovery of the quantum sl_2 invariant of links

We put $X = \sqcup^l S^1$, and fix an order of components of X. Then, for a Lie algebra $\mathbf{g}$ and its representations $R_1, \dots, R_l$, we have a linear map $W_{\mathbf{g}; R_1, \dots, R_l} : \mathscr{A} \to \mathbb{C}$, defined by substituting $\mathbf{g}$ to dashed lines of chord diagrams and substituting $R_1, \dots, R_l$ to l solid circles of chord diagrams; see [1] for a detailed definition of the map. We call the map *the weight system* derived from the substitution of $\mathbf{g}$ and $R_1, \dots, R_l$. If X is not closed, $W_{\mathbf{g}; R_1, \dots, R_l}(D)$ becomes an intertwiner for $D \in \mathscr{A}(X)$. For example, $W_{\mathbf{g}; R_1, R_2}($  $)$ is an intertwiner of $R_1 \otimes R_2$ to itself.

For $X = \sqcup^l S^1$, we also define *the modified weight system* $\hat{W}_{\mathbf{g}; R_1, \dots, R_l} : \mathscr{A}(X) \to \mathbb{C}[[h]]$ by

$$\hat{W}_{\mathbf{g}; R_1, \dots, R_l}(D) = \hat{W}_{\mathbf{g}; R_1, \dots, R_l}(D) h^{deg(D)}$$

where $\mathbb{C}[[h]]$ denotes the power series ring in the indeterminate h, and $deg(D)$ denotes the degree of D, i.e., half times the number of the univalent and trivalent vertices of the chord diagram D.

Let q be an indeterminate. We denote by $Q^{\mathbf{g};R_1,\ldots,R_l}(L)\in\mathbb{Z}[q^{1/2N},q^{-1/2N}]$, where N is the rank of $\mathbf{g}$, the quantum $(\mathbf{g};R_1,\ldots,R_l)$ invariant of an oriented framed link L; see for example [8, 36] for its definition.

The following theorem implies the universality of the modified Kontsevich invariant $\hat{Z}(L)$ of a framed link L among the quantum invariants.

Theorem 1.2 (Kassel [7, Theorem XX.8.3] and Le and Murakami [20, Theorem 10]). *Let L be a framed link, $\mathbf{g}$ a Lie algebra and $R_1,\ldots,R_l$ representations of $\mathbf{g}$. Then the quantum $(\mathbf{g};R_1,\ldots,R_l)$ invariant recovers from the Kontsevich invariant through the modified weight system $\hat{W}_{\mathbf{g};R_1,\ldots,R_l}$ as*

$$\hat{W}_{\mathbf{g};R_1,\ldots,R_l}(\hat{Z}(L))=Q^{\mathbf{g};R_1,\ldots,R_l}(L)|_{q=e^h}.$$

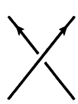
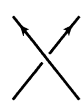
We have a conceptual proof of the theorem using uniqueness of category of quasi-Hopf algebras, see [7, Theorem XX.8.3] and [20, Theorem 10]. Further, in the case $\mathbf{g}=sl_2$, we have a direct proof of the theorem using the skein relation due to Jun Murakami and Thang Le [20], in which they showed similar results for the quantum sl_N invariant of links without framings. Here, we give a modified proof below for the quantum sl_2 invariant of framed links along their idea.

Proof of Theorem 1.2 for $\mathbf{g}=sl_2$. Let V_m be the m-dimensional irreducible representation of the Lie algebra sl_2. We show

$$\hat{W}_{sl_2;V_{m_1},\ldots,V_{m_l}}(\hat{Z}(L))=Q^{sl_2;V_{m_1},\ldots,V_{m_l}}(L)|_{q=e^h}$$

for any framed link L and any integers $m_1,\ldots,m_l$.

Step 1. *The case $m_1=\cdots=m_l=2$:* The quantum invariant $Q^{sl_2;V_2,\ldots,V_2}(L)$ is uniquely characterized by the following three formulas and the property that it is multiplicative with respect to the split union of two links;

$q^{1/4}Q^{sl_2;V_2,\ldots,V_2}($  $)-q^{-1/4}Q^{sl_2;V_2,\ldots,V_2}($  $)=(q^{1/2}-q^{-1/2})Q^{sl_2;V_2,\ldots,V_2}($  $)$, (1.1)

$Q^{sl_2;V_2,\ldots,V_2}($  $)=q^{3/4}Q^{sl_2;V_2,\ldots,V_2}($  $)$ (1.2)

$$Q^{sl_2;V_2}(O)=[2]=q^{1/2}+q^{-1/2} \tag{1.3}$$

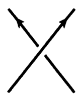
where the first formula implies a relation between the values of the three links which are locally different from each other as shown in the pictures in the formula; we often call such a relation a *skein relation*. Further, the second formula implies that, when the framing of a component of a link increases by 1, the invariant becomes $q^{3/4}$ times the invariant of the original link, where in pictures we show the framing of a link by the blackboard framing. In the third formula, O denotes the trivial knot with 0 framing and we put the q-integer by $[m]=(q^{m/2}-q^{-m/2})/(q^{1/2}-q^{-1/2})$.

It is sufficient to show that $\hat{W}_{sl_2;V_2,\dots,V_2}(\hat{Z}(L))$ also satisfies the three relations.

Firstly, we show that $\hat{W}_{sl_2;V_2,\dots,V_2}(\hat{Z}(L))$ satisfies the same relation as (1.1). We denote the three pictures in the formula in Lemma 1.3 below by H, 1 and P respectively, i.e., we have

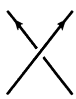
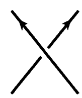
$$W_{sl_2;V_2,V_2}(H) = -\frac{1}{2}\,W_{sl_2;V_2,\dots,V_2}(1) + W_{sl_2;V_2,V_2}(P).$$

Hence, we have

$e^{h/4}\hat{W}_{sl_2;V_2,V_2}(\hat{Z}($  $)) = e^{h/4}W_{sl_2;V_2,V_2}\left(P\exp\left(\frac{h}{2}H\right)\right)$

$$= W_{sl_2;V_2,V_2}\left(P\exp\left(\frac{h}{2}P\right)\right).$$

Therefore, noting the relation $P^2 = 1$, we have

$e^{h/4}\hat{W}_{sl_2;V_2,V_2}(\hat{Z}($  $)) - e^{-h/4}\hat{W}_{sl_2;V_2,V_2}(\hat{Z}($  $))$

$$= W_{sl_2;V_2,V_2}\left(P\exp\left(\frac{h}{2}P\right) - P\exp\left(-\frac{h}{2}P\right)\right)$$

$$= W_{sl_2;V_2,V_2}\left(\exp\left(\frac{h}{2}\right) - \exp\left(-\frac{h}{2}\right)\right)$$

$$= (e^{h/2} - e^{-h/2})W_{sl_2;V_2,V_2}(1)$$

which implies (1.1) for $\hat{W}_{sl_2;V_2,\dots,V_2}(\hat{Z}(L))$, completing this case.

Secondly, we show that $\hat{W}_{sl_2;V_2,\dots,V_2}(\hat{Z}(L))$ satisfies the same relation as (1.2). As in [25] we have

$\hat{Z}($  $) = \hat{Z}($  $)\,\#\,e^{\Theta/2}$,

where Θ denotes the chord diagram consisting of a solid circle and a dashed arc. Hence, we obtain the same relation as (1.2) for $\check{Z}(L)$ as

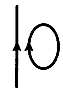
$\hat{W}_{sl_2;V_2}(\hat{Z}($  $)) = e^{3h/4}\hat{W}_{sl_2;V_2}(\hat{Z}($  $))$

using the following formula:

$W_{sl_2;V_2}($  $) = \frac{3}{2}\,W_{sl_2;V_2}($  $)$. (1.4)

Here we obtain (1.4) as follows. By definition of the weight system, one isolated dashed chord is related, through $W_{sl_2;V_2}$, to the eigenvalue of Casimir element of sl_2 on V_2. Since the eigenvalue is equal to 3/2, we obtain (1.4).

Lastly, we show that $\hat{W}_{sl_2;V_2,\dots,V_2}(\hat{Z}(L))$ satisfies the same relation as (1.3). By using the results proved above, we have

$(q^{1/2}-q^{-1/2})\hat{W}_{sl_2;V_2,V_2}(\hat{Z}($  $)) = q^{1/4}\hat{W}_{sl_2;V_2}(\hat{Z}($  $)) - q^{-1/4}\hat{W}_{sl_2;V_2}(\hat{Z}($  $))$

$= (q-q^{-1})\hat{W}_{sl_2;V_2}(\hat{Z}($  $)).$

This implies (1.3) for $\hat{W}_{sl_2;V_2,\dots,V_2}(\hat{Z}(L))$.

Step 2. *The case that one of m_j's is greater than* 2: For simplicity, we assume that L is a framed knot. Then $\hat{Z}(L)$ belongs to $\mathscr{A}(S^1)$.

Firstly, we show $\hat{W}_{sl_2;V_1}(\hat{Z}(L)) = Q^{sl_2;V_2}(L)|_{q=e^h}$. The left-hand side is equal to $\hat{W}_{sl_2}(\varepsilon(\hat{Z}(L)))$ by definition of the map ε. Further, by Proposition 1.1, it is equal to $\hat{W}_{sl_2}(\hat{Z}(\phi))$ for the empty link ϕ. Furthermore it is equal to 1, since $\hat{Z}(\phi)=1$. On the other hand, the right-hand side is equal to 1 by definition of the quantum invariant, since V_1 is the trivial representation. Hence, we complete this case.

Secondly, we have $\hat{W}_{sl_2;V_2}(\hat{Z}(L)) = Q^{sl_2;V_2}(L)|_{q=e^h}$ by Step 1.

Lastly, we show $\hat{W}_{sl_2;V_m}(\hat{Z}(L)) = Q^{sl_2;V_m}(L)|_{q=e^h}$ by induction on m. By definition of Δ, we have

$$\hat{W}_{sl_2;V_{m-1},V_2}(\Delta(\hat{Z}(L))) = \hat{W}_{sl_2;V_{m-1}\otimes V_2}(\hat{Z}(L)).$$

Since $\Delta(\hat{Z}(L)) = \hat{Z}(L^{(2)})$ by Proposition 1.1 where $L^{(2)}$ denotes the 2-parallel of L, we have

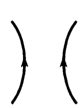
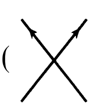
$$\begin{aligned}\hat{W}_{sl_2;V_{m-1},V_2}(\hat{Z}(L^{(2)})) &= \hat{W}_{sl_2;V_{m-1}\otimes V_2}(\hat{Z}(L))\\ &= \hat{W}_{sl_2;V_{m-2}\oplus V_m}(\hat{Z}(L))\\ &= \hat{W}_{sl_2;V_{m-2}}(\hat{Z}(L)) + \hat{W}_{sl_2;V_m}(\hat{Z}(L))\end{aligned}$$

where the last equality is derived from the definition of the weight system. Further, the same property holds for quantum invariants

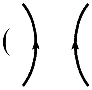
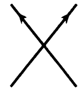
$$Q^{sl_2;V_{m-1},V_2}(L^{(2)}) = Q^{sl_2;V_{m-2}}(L) + Q^{sl_2;V_m}(L)$$

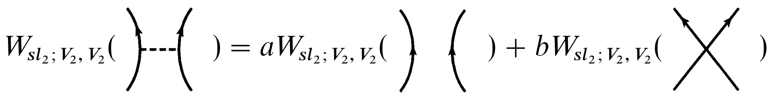
by definition of quantum invariants, since $V_{m-1}\otimes V_2 = V_{m-2}\oplus V_m$ also holds for representations of the quantum group $U_q(sl_2)$. By reducing the proof to the case of $m-1$ and $m-2$,[1] we obtain the required formula by induction on m. $\square$

Lemma 1.3.

$W_{sl_2;V_2,V_2}($  $) = -\frac{1}{2}W_{sl_2;V_2,V_2}($  $) + W_{sl_2;V_2,V_2}($  $)$

[1] To be precise, we should have reduced the case of m for l-component links to the case of $m-2$ for l component links and the case of $m-1$ for $(l+1)$-component links, by proving the theorem for any positive integer l at the same time, though we show the proof only for the case $l=1$ to avoid complexity of subscripts in formulas in the proof.

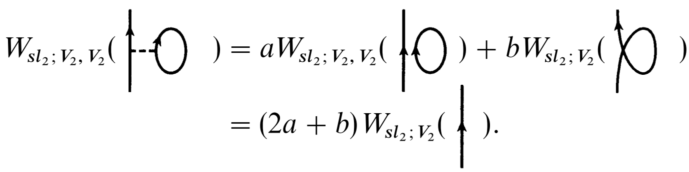
Proof. Since $V_2 \otimes V_2$ is isomorphic to $V_1 \oplus V_3$, the set of intertwiners $\mathrm{Hom}_{sl_2}(V_2 \otimes V_2)$ is two dimensional. Noting that $W_{sl_2;V_2,V_2}($  $)$ and $W_{sl_2;V_2,V_2}($  $)$ are linearly independent in the set, we put



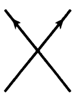
(1.5)

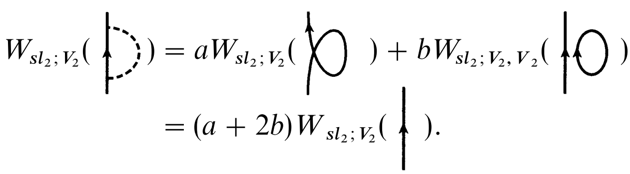
with undetermined coefficients a and b.

By taking closure of the two right ends of each diagrams in (1.5), we have



Since the left-hand side vanishes by definition of the weight system, we have $2a + b = 0$.

By gluing  to the bottom of (1.5) and taking closure of the right two ends after that, we have

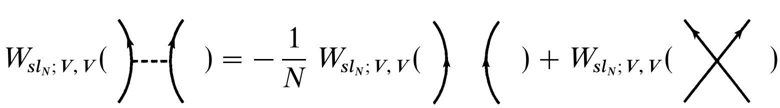


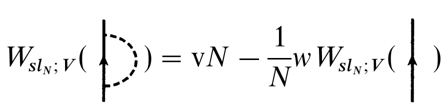
Hence by (1.4) we have $a + 2b = 3/2$. With the relation $2a + b = 0$, we obtain $a = -1/2$ and $b = 1$. $\square$

Remark 1.4. We can obtain a similar result as Step 1 in the proof of Theorem 1.2 between $\hat{Z}(L)$ and the quantum $(sl_N; V, \dots, V)$ invariant of framed links for the vector representation V of sl_N, i.e., we have

$$\hat{W}_{sl_N;V,\dots,V}(\hat{Z}(L)) = Q^{sl_N;V,\dots,V}(L)|_{q=e^h}$$

by using





which are obtained in the same way as Step 1 in the proof of Theorem 1.2.

We have the following corollary of Theorem 1.2.

Corollary 1.5 (Le and Murakami [21]).

$$\hat{W}_{sl_2;V_m}(v) = \frac{[m]}{m} \in Hom_{sl_2}(V_m, V_m).$$

Proof. Let O be the trivial knot with 0 framing. Since $\hat{Z}(O)$ is equal to the closure of v, we have

$$\hat{W}_{sl_2;V_m}(\hat{Z}(O)) = (\dim V_m) \cdot \hat{W}_{sl_2;V_m}(l) = m\hat{W}_{sl_2;V_m}(v).$$

On the other hand, by Theorem 1.2, we have

$$\hat{W}_{sl_2;V_m}(\hat{Z}(O)) = Q^{sl_2;V_m}(O) = [m].$$

These formulas imply the required formula. □

2. Recovery of the perturbative $SO(3)$ invariant

In this section we show the recovery of the perturbative $SO(3)$ invariant $q^{SO(3)}(M)$ from the universal perturbative invariant $\hat{\Omega}(M)$ for rational homology 3-spheres M.

2.1. Replacing j_n by a power series of α

Fixing a closed component C of X, we define the map $\alpha : \mathscr{A}(X) \to \mathscr{A}(X)$ by $\alpha(D) = \Delta'_{(C)}(D) - u \cdot D$, where $\Delta'_{(C)}(D)$ denotes the element obtained from $\Delta_{(C)}(D)$ by replacing one of the copies of C with a dashed line; note that $\Delta_{(C)}(D)$ includes two copies of C. Further, u denotes a dashed loop and the product implies the disjoint union of chord diagrams. Note that, though $\mathscr{A}(X)$ does not allow dashed loops, the map a is a well-defined operation on $\mathscr{A}(X)$. We show some simple cases in Fig. 2.

By definition of α, the number of dashed univalent vertices on C decreases by at least 2 by the map α, where we apply the map a keeping the dashed uni-trivalent graph unchanged, not using the STU relation; see Fig. 2 for simple cases. Hence we have

Lemma 2.1. *If the number of dashed univalent vertices of a chord diagram D on C is less than* $2k$, *then* $\alpha^k(D) = 0$, *where* α^k *denotes the composition of* k *copies of* α.

As in [25], we define the map $j_n : \mathscr{A}(\sqcup^l S^1) \to \mathscr{A}(\phi)$ by replacing each solid circle with m dashed univalent vertices by the dashed graph T^n_m given in [25], where $\mathscr{A}(\phi)$ denotes the space of chord diagrams on the empty 1-manifold, i.e., the vector space spanned by dashed trivalent graphs subject to the AS and IHX relations. (To be precise the space $\mathscr{A}(\phi)$ as the image of the map j_n includes dashes loops.) To investigate a relation between j_n and α, we redefine the map j_n as the map $\mathscr{A}(S^1, k) \to \mathscr{A}(k)$ by replacing the solid circle S^1 with m dashed univalent vertices by T^n_m, where $\mathscr{A}(S^1, k)$ denotes the space of chord diagrams on S^1 with k external dashed univalent vertices which are not on S^1, and $\mathscr{A}(k)$ denotes the space of chord diagrams on the empty solid set with k external univalent vertices, i.e., dashed univalent graphs with k univalent vertices. As usual, chord diagrams

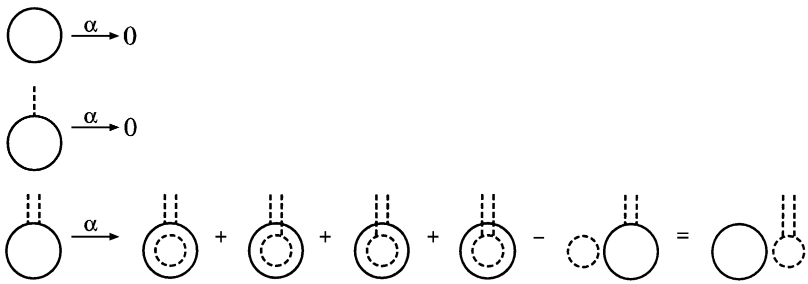


Fig. 2. Definition of α for simple cases.

are subject to the AS, IHX and STU relations in the spaces. We also redefine the map α as a map $\mathscr{A}(S^1, k) \to \mathscr{A}(S^1, k)$ by $a(D) = \Delta'_{(S^1)}(D) - u \cdot D$ as above.

We can compute the map j_n using the map α by the following proposition.

Proposition 2.2. *Let $D_m \in \mathscr{A}(S^1, m)$ be the chord diagram consisting of a solid circle and m dashed lines such that a univalent vertex of each of m dashed lines is on the solid circle. Then, there exists a power series $p(\alpha) = \sum_{i=1}^{\infty} c_i a^i$ with coefficients $c_i \in \mathbb{Z}[1/2, 1/3, \cdots, 1/(2i+1)]$[2] satisfying the formula*

$$W_{sl_2}(j_1(D_m)) = W_{sl_2}((\varepsilon \circ p(a))(D_m))$$

for any non-negative integer m. Here $p(\alpha)$ is a map by regarding α^i as the composition of i copies of $\alpha : \mathscr{A}(S^1, m) \to \mathscr{A}(S^1, m)$.

Proof. For any power series $p(\alpha) = \sum_{i=1}^{\infty} c_i \alpha^i$, by Lemma 2.1, we have

$$p(\alpha)(D_m) = p_k(\alpha)(D_m)$$

for each $m \leqslant 2k$, where we put $p_k(a) = \sum_{i=1}^{k} c_i a^i$. Hence it is sufficient to show the existence of an infinite series of scalars $c_1, c_2, c_3, \ldots$ satisfying

$$W_{sl_2}(j_1(D_m)) = W_{sl_2}((\varepsilon \circ p_k(\alpha))(D_m)) \tag{2.1}$$

for each k and for each $m \leqslant 2k$. We show (2.1) by induction on k as follows.

Suppose that (2.1) holds for $k-1$, i.e., there exists a finite series $c_1, \cdots, c_{k-1}$ satisfying

$$W_{sl_2}(j_1(D_m)) = W_{sl_2}((\varepsilon \circ p_{k-1}(\alpha))(D_m)), \tag{2.2}$$

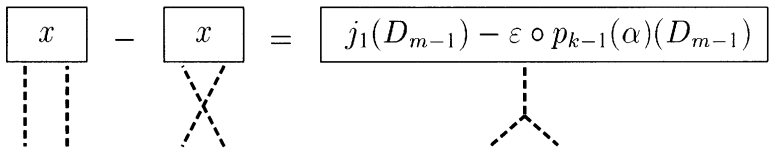
for each $m \leqslant 2k-2$. Then, by Lemma 2.1, the required formula (2.1) holds for $m \leqslant 2k-2$, even if we put c_k to be any value.

[2] Recently, the concrete formula of c_i's has been obtained as $c_i = (-1)^{i+1}((i-1)!)^2/2(2i)!$ by Dylan Thurston and Thang Le (in private communication).

Further we show that (2.1) holds for $m = 2k - 1$ as follows; note that the right-hand side does not still depend on a choice of c_k, as in the above case, by Lemma 2.1. Put

$$x = j_1(D_m) - (\varepsilon \circ p_{k-1}(\alpha))(D_m). \tag{2.3}$$

Then $W_{sl_2}(x)$ belongs to $(sl_2)^{\otimes m}$. By interchanging two adjacent dashed ends, we have



where we used the STU relation on two adjacent dashed ends of D_m. Since the right-hand side in the above formula vanishes by the hypothesis of induction, $W_{sl_2}(x)$ is invariant under the change of two adjacent dashed ends. Hence it is further invariant under any change of dashed ends. Therefore $W_{sl_2}(x)$ belongs to the invariant space $((sl_2)^{\otimes m})^{\mathscr{S}_m}$, where the symmetric group $\mathscr{S}_m$ acts on $(sl_2)^{\otimes m}$ by changing entries. Further, by regarding $W_{sl_2}(x)$ as the image of $1 \in \mathbb{C}$ by an intertwiner,[3] $W_{sl_2}(x)$ belongs to $((sl_2)^{\otimes m})^{sl_2}$, where sl_2 acts on $(sl_2)^{\otimes m}$ as the tensor product of m copies of the adjoint representation of sl_2. Therefore $W_{sl_2}(x)$ belongs to the invariant space $((sl_2)^{\otimes m})^{\mathscr{S}_m, sl_2}$. The invariant space is one dimensional if m is even, and is the null vector space if m is odd, by invariant theory; for example, see [6]. Since m is odd in this case, $W_{sl_2}(x)$ vanishes, because it belongs to the null vector space. Hence, by (2.3), we obtain (2.1) for $m = 2k - 1$, since we have $(p_{k-1}(\alpha))(D_m) = (p_k(\alpha))(D_m)$ by Lemma 2.1.

We show that (2.1) holds for $m = 2k$ for a suitably chosen c_k as follows. We put x as in (2.3) again, and repeat the same argument as above. In this case $W_{sl_2}(x)$ belongs to one dimensional vector space, since m is even. Further $W_{sl_2}((\varepsilon \circ \alpha^k)(D_m))$ is non-zero in the space; in fact, we see below that it does not vanish, when dashed ends are closed. Hence we put

$$W_{sl_2}(x) = c_k W_{sl_2}((\varepsilon \circ a^k)\,(D_m))$$

for some scalar c_k. Therefore we have

$$\begin{aligned} W_{sl_2}(j_1(D_m)) &= W_{sl_2}((\varepsilon \circ p_{k-1}(\alpha))(D_m) + x) \\ &= W_{sl_2}((\varepsilon \circ (p_{k-1}(\alpha) + c_k \alpha^k))(D_m)). \end{aligned}$$

Putting $p_k(\alpha) = p_{k-1}(\alpha) + c_k \alpha^k$, we obtain (2.1) for $m = 2k$.

We evaluate the factors of the denominator of c_k by induction on k, as follows. Let Θ^k be the chord diagram consisting of a solid circle with k isolated dashed chords. Since Θ^k has $2k$ dashed chords on S^1, we have

$$W_{sl_2}(j_1(\Theta^k)) = W_{sl_2}(\varepsilon \circ p_k(\alpha)(\Theta^k)).$$

[3] Here we mean by $\mathbb{C}$ the trivial representation of sl_2.

Hence we have

$$c_k W_{sl_2}((\varepsilon \circ \alpha^k)(\Theta^k)) = W_{sl_2}(j_1(\Theta^k)) - \sum_{i=1}^{k-1} c_i W_{sl_2}((\varepsilon \circ \alpha^i)(\Theta^k)). \tag{2.4}$$

By definition of j_1, $j_1(\Theta^k)$ is equal to the chord diagram obtained from the tree T_{2k} (given in [25]) by closing $2k$ ends with k isolated chords. Hence the first term in the right hand side of (2.4) belongs to $\mathbb{Z}[1/2, 1/3, \ldots, 1/(2k-1)]$. Further the second terms belong to $\mathbb{Z}[1/2, 1/3, \ldots, 1/(2k-3)]$ by the hypothesis of induction. On the other hand, we calculate the left-hand side as follows. $(\varepsilon \circ \alpha^k)(\Theta^k)$ consists of terms such that each of k α's decreases exactly two chord of Θ^k; note that the other terms vanish. Each α makes a dashed loop with two dashed segments. W_{sl_2} takes it to 4. Hence we have

$$W_{sl_2}((\varepsilon \circ \alpha)(\Theta^k)) = 4^k W_{sl_2}(j_k(\Theta^k)) = 4^k(2k+1)!!.$$

By (9), we obtain $c_k \in \mathbb{Z}[1/2, 1/3, \ldots, 1/(2k+1)]$. □

Proposition 2.3. *Under the assumption of Proposition* 2.2,

$$W_{sl_2}(j_n(D_m)) = \frac{1}{n!} W_{sl_2}((\varepsilon \circ p(\alpha)^n)(D_m))$$

holds for any positive integer n.

Proof. By definition of Δ, we have $(\Delta^{(k_1)} \otimes \Delta^{(k_2)}) \circ \Delta = \Delta^{(k_1+k_2+1)}$. Since $\alpha^i(D)$ is a linear sum of chord diagrams $\Delta^{(k)}(D)$ possibly replaced some solid circles with dashed ones, we have the following formula by the above formula,

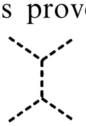
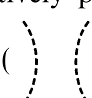
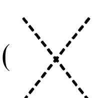
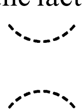
$$((\varepsilon \circ p_1(\alpha)) \otimes (\varepsilon \circ p_2(\alpha))) \circ \Delta = \varepsilon \circ (p_1(\alpha) p_2(\alpha)) \tag{2.5}$$

for any two power series $p_1(\alpha)$ and $p_2(\alpha)$, where $p_1(\alpha)p_2(\alpha)$ implies the usual product as power series.

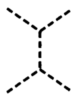
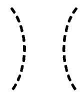
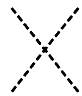
Further, since $j_n = (1/n!) j_1 \circ \Delta^{(n-1)}$ holds by definition of j_n, we have

$$W_{sl_2}(j_n(D_m)) = \frac{1}{n!} W_{sl_2}(((\varepsilon \circ p(\alpha))^{\otimes n} \circ \Delta^{(n-1)})(D_m))$$

by Proposition 2.2, noting that the solid circle in D_m becomes n solid circles by $\Delta^{(n-1)}$. By applying the formula (2.5) $n-1$ times, we obtain the required formula. □

The following lemma is proved in [4, Theorem 6]. We can alternatively prove the lemma by using the fact that $W_{sl_2}($  $)$ can be expressed as a linear sum of $W_{sl_2}($  $)$, $W_{sl_2}($  $)$ and $W_{sl_2}($  $)$ in the same way as in the proof of Lemma 1.3.

Lemma 2.4 (Chmutov and Varchenko [4]).

$W_{sl_2}($  $) = 2(W_{sl_2}($  $) - W_{sl_2}($  $))$.

2.2. Calculations of representations modulo a prime

Recall that V_m is the m dimensional irreducible representation of sl_2. We denote by $R(sl_2)$ the subring of the representation ring of sl_2 generated by V_m with odd m. Note that the ring $R(sl_2)$ is isomorphic to the polynomial ring $\mathbb{Z}[V_3]$. We put $a \in R(sl_2)$ to be $V_3 - 3V_1$.

Lemma 2.5. *Let r be an odd prime.*

1. *The following formula holds*:

$$-2a^{(r-3)/2} = \sum_{\substack{1 \leqslant m \leqslant r-2 \\ \text{m is odd}}} mV_m + rR$$

for some $R \in R(sl_2)$, *such that* R *is a linear sum of* $V_1, V_3, V_5, \ldots, V_{r-2}$.

2. *For each* $k > (r-3)/2$, *the following formula holds*:

$$a^k = rR + V_r \otimes R'$$

for some $R, R' \in R(sl_2)$ *satisfying the same condition on* R *as above.*

Proof. To obtain formula (2.8) below for $a^k \in R(sl_2)$, we extend the notation V_m as follows. For negative integer m, we define V_m to be $-V_{-m} \in R(sl_2)$. Then the following ordinary formula[4] for the decomposition of a tensor product

$$V_m \otimes V_3 = V_{m-2} \oplus V_m \oplus V_{m+2}$$

also holds for each (even negative) odd integer m by definition of V_m. Further, by definition of a, we have

$$V_m a = V_{m-2} - 2V_m + V_{m+2} \in R(sl_2). \tag{2.6}$$

We consider the injection i: $\text{span}_Z\{V_m\} \to \mathbb{Z}[t, t^{-1}]$ defined by $i(V_m) = t^m$ such that it is a homomorphism between $\mathbb{Z}$-modules. Formula (2.6) implies

$$i(V_m a) = (t^{-2} - 2 + t^2)i(V_m).$$

By applying this formula k times to V_1, we have

$$i(a^k) = (t^{-2} - 2 + t^2)^k t = (t - t^{-1})^{2k} t. \tag{2.7}$$

[4] See, for example, [6].

We put the right-hand side of (2.7) to be $f(t)$. Formula (2.7) implies that, when expanding a^k as a linear sum of V_m for positive and negative integers m, the coefficient of V_m in the linear sum is equal to the coefficient of t^m in $f(t)$. Since we put $V_m = -V_{-m}$, when expanding a^k as a linear sum of V_m for only positive integers m, the coefficient of V_m in the linear sum is equal to

$$\begin{aligned}&(\text{the coefficient of } t^m \text{ in } f(t)) - (\text{the coefficient of } t^{-m} \text{ in } f(t))\\ &\quad = (\text{the coefficient of } t^m \text{ in } f(t)) - (\text{the coefficient of } t^m \text{ in } f(t^{-1}))\\ &\quad = (\text{the coefficient of } t^m \text{ in } (t-t^{-1})^{2k}t - (t-t^{-1})^{2k}t^{-1}).\end{aligned}$$

The polynomial in t in the last line is equal to

$$(t-t^{-1})^{2k+1} = \sum_{\substack{|m| \leqslant 2k+1 \\ m \text{ is odd}}} (-1)^{k+(m-1)/2} \binom{2k+1}{k+(m+1)/2} t^m.$$

Hence we have

$$a^k = \sum_{\substack{1 \leqslant m \leqslant 2k+1 \\ \text{m is odd}}} (-1)^{k+(m-1)/2} \binom{2k+1}{k+(m+1)/2} V_m. \tag{2.8}$$

If $k=(r-3)/2$, then by (2.8) the coefficient of V_m is congruent modulo r to

$$\begin{aligned}&(-1)^{(r+m-4)/2}\binom{r-2}{(r+m-2)/2}\\ &\quad \equiv (-1)^{(r+m-4)/2}\frac{(-2)(-3)\cdots(-(r+m)/2)}{((r+m-2)/2)!} = -\frac{r+m}{2} \equiv -\frac{m}{2}.\end{aligned}$$

This implies (1) of the lemma.

If $k=(r-1)/2$, then by (2.8) the coefficient of V_m is equal to

$$(-1)^{(r+m-2)/2}\binom{r}{(r+m)/2}$$

which is divisible by r for $m<r$. Hence we obtain (2) for $k=(r-1)/2$.

If $k>(r-1)/2$, then we show (2) by induction on k as follows. Suppose a^{k-1} is expressed as $rR + V_r \otimes R'$ such that R is equal to a linear sum of $V_1, V_3, \ldots, V_{r-2}$. If the linear sum does not contain V_{r-2}, then (2) also holds for a^k by putting Ra and $R'a$ to be new R and R'. If the linear sum contains V_{r-2}, we obtain (2) modifying the above case, noting that $V_{r-2}a$ is equal to a linear sum of V_{r-4}, V_{r-2} and V_r. □

We can replace the map a with a by the following lemma.

Lemma 2.6. *For any $D \in \mathscr{A}(\sqcup^l S^1)$, the following formula holds:*

$$W_{sl_2}(((\varepsilon\circ\alpha^{n_1})\otimes(\varepsilon\circ\alpha^{n_2})\otimes\cdots\otimes(\varepsilon\circ\alpha^{n_l}))(D)) = W_{sl_2;a^{n_1},a^{n_2},\ldots,a^{n_l}}(D).$$

Proof. For simplicity, we show the proof for $l = 1$. By definition of ε, we have

$$W_{sl_2}(\varepsilon(D)) = W_{sl_2;V_1}(D) \tag{2.9}$$

for any $D \in \mathscr{A}(S^1)$. Further, by definition of Δ, we have

$$W_{sl_2;R_1,R_2}(\Delta(D)) = W_{sl_2;R_1,R_2}(D)$$

for any $D \in \mathscr{A}(S^1)$. In the definition of α, we replace one solid line with dashed line. After that, we substitute sl_2 into the dashed line. We replace the procedure by substitution of the adjoint representation of sl_2, which is isomorphic to V_3 in our notation, into the solid line. Then, by the definition of α, we have

$$W_{sl_2;R}(\alpha(D)) = W_{sl_2;aR}(D). \tag{2.10}$$

Applying (2.9) and (2.10) necessary times, we obtain the required formula. □

2.3. *Recovery of the perturbative $SO(3)$ invariant*

Let M be a rational homology 3-sphere throughout this section. In this section we prove recovery of the perturbative $SO(3)$ invariant $\tau^{SO(3)}(M)$ from the universal perturbative invariant $\hat{\Omega}(M)$; the former invariant is defined only for rational homology 3-spheres, while the latter invariant is defined for any closed oriented 3-manifolds.

We review the definition of $\tau^{SO(3)}(M)$ in [29], which we call the perturbative $SO(3)$ invariant of M. The invariant $q^{SO(3)}(M)$ is defined by the following characterization; it determines the invariant uniquely. The quantum $SO(3)$ invariant $q_r^{SO(3)}(M)$ defined by Kirby and Melvin [9] belongs to $\mathbb{Z}[\zeta]$ for any odd prime r, see [27], where we put $\zeta = \exp(2\pi\sqrt{-1}/r)$. By expanding the invariant in power series in $\zeta - 1$, we obtain scalar invariants $\lambda_{r,n}$:

$$\tau_r^{SO(3)}(M) = \left(\frac{|H_1(M;\mathbb{Z})|}{r}\right) \sum_n \lambda_{r,n}(\zeta - 1)^n$$

where $(\frac{\cdot}{r})$ denotes the Legendre symbol, noting that the invariant $\lambda_{r,n}$ is well defined in $\mathbb{Z}/r\mathbb{Z}$ for $n \leqslant r-2$. The author showed in [28, 29] that there exists the series of invariants $\lambda_n \in \mathbb{Z}[1/2, 1/3, \ldots, 1/(2n+1)]$ such that λ_n is congruent to $\lambda_{r,n}$ modulo r for $n \leqslant (r-3)/2$, and defined the invariant $\tau^{SO(3)}(M)$ to be $\sum_{n=0}^{\infty} \lambda_n (q-1)^n$. Modifying the definition, we define the *perturbative $SO(3)$ invariant* by

$$\tau^{SO(3)}(M) = \sum_{n=0}^{\infty} \lambda_n (e^h - 1)^n \in \mathbb{Q}[[h]] \tag{2.11}$$

in the present paper. By the definition, we characterize the invariant by

$$(\tau^{SO(3)}(M)|_{h=\log(s+1)}{}^{(\leqslant (r-3)/2)})|_{s=\zeta-1}$$
$$= \left(\frac{|H_1(M;\mathbb{Z})|}{r}\right) \tau_r^{SO(3)}(M) + O((\zeta-1)^{(r-1)/2}; \mathbb{Z}_{(r)}[\zeta]), \tag{2.12}$$

where $\mathbb{Z}_{(r)}$ denotes $\mathbb{Z}[1/2, 1/3, \ldots, 1/(r-1)]$. Here, the first subscript on the left-hand side implies the substitution of

$$\log(s+1) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{s^k}{k} \in \mathbb{Q}[[s]] \tag{2.13}$$

into h, and the power $(\leqslant (r-3)/2)$ implies removing the terms of degree $> (r-3)/2$ in $\mathbb{Q}[[s]]$. Further, the last $O(\cdots)$ denotes the remaining terms such that, for a ring $\mathscr{R}$, $\cdots + O(x^k; \mathscr{R})$ implies $\cdots + ux^k$ for some $u \in \mathscr{R}$.

We also briefly review the definition of the universal perturbative invariant $\hat{\Omega}(M)$ in [25]. Let M be a rational homology 3-sphere obtained from S^3 by integral surgery along a l-component framed link L in S^3. For the modified Kontsevich invariant $\hat{Z}(L)$, we put $\check{Z}(L)$ to be $\hat{Z}(L) \# v^{\otimes l}$ which is obtained from $\hat{Z}(L)$ by taking connected sum of the element v (see Corollary 1.5) to each component of L. For each positive integer n, the degree $\leqslant n$ part of *the universal perturbative invariant* $\hat{\Omega}(M)$ is defined to be the degree $\leqslant n$ part of

$$(\iota_n \check{Z}(U_+))^{-\sigma_+} (\iota_n \check{Z}(U_-))^{-\sigma_-} \iota_n \check{Z}(L)$$

where $U_\pm$ are trivial knots with ± 1 framings, $\sigma_\pm$ are the positive and negative eigenvalues of the linking matrix of L and ι_n is the map such that $\iota_n(D)$ is obtained from $j_n(D)$ (given in Section 2.1) by replacing each dashed loop with $-2n$; see [25] for detailed definition.

The following theorem is the main theorem of this paper.

Theorem 2.7. *For a rational homology* 3-*sphere* M,

$$\tau^{SO(3)}(M) = \frac{1}{|H_1(M;\mathbb{Z})|} \hat{W}_{sl_2}(\hat{\Omega}(M))$$

holds, i.e. the perturbative $SO(3)$ *invariant of rational homology* 3-*spheres recovers from the universal perturbative invariant by substituting* sl_2 *into chord diagrams.*

Before showing a precise proof of the theorem, we show a sketch of the proof. Note that some formulas in the sketch lack exactness to avoid complexity of the argument, mainly on integrality of the coefficients of the formulas. We deal with such complexity precisely in the proof shown after the sketch.

Proof (*Sketch*)**.** For simplicity, suppose that M is obtained from S^3 by Dehn surgery along a framed knot K with a positive framing. We replace j_n by the power series of a by Proposition 2.3 as

$$j_n(\check{Z}(K)) = \frac{1}{n!} (\varepsilon \circ p(\alpha)^n)(\check{Z}(K))$$

$$= \sum_{i \geqslant n} c_{i;n} (\varepsilon \circ \alpha^i)(\check{Z}(K))$$

where we put $p(\alpha)^n/n! = \sum_{i=n}^{\infty} c_{i;n} \alpha^i$.

We restrict our attention to terms of at most a suitably fixed degree.[5] By Lemma 2.1 which is a vanishing lemma for α, we can reduce the right-hand side of the above formula to a finite sum as follows:

$$j_n(\check{Z}(K)) = \sum_{i \geqslant n}^{\text{finite}} c_{i;n} (\varepsilon \circ \alpha^i)(\check{Z}(K)) + (\text{terms of high degrees}).$$

For simplicity, we omit the comment of "high degrees" in the following of this sketch.

We replace α^i with a^i by applying Lemma 2.6, to obtain

$$h^n \cdot \hat{W}_{sl_2}(j_n(\check{Z}(L))) = \sum_{i \geqslant n}^{\text{finite}} c_{i;n} \hat{W}_{sl_2;a^i}(\check{Z}(K)). \tag{2.14}$$

By Lemma 2.5 which is a vanishing lemma for a, we have[6]

$$a^k(r) \underset{(r)}{=} \begin{cases} -\frac{1}{2}\sum m V_m & \text{if } k = \frac{r-3}{2} \\ 0 & \text{if } k > \frac{r-3}{2} \end{cases} \tag{2.15}$$

Put $n = (r-3)/2$, then terms in the right-hand side of (19) vanish modulo r except for the first term, that is, we obtain[7]

$$\begin{aligned} h^n \cdot \hat{W}_{sl_2}(j_n(\check{Z}(K))) &\underset{(r)}{=} -\frac{c_{n;n}}{2} \hat{W}_{sl_2;\sum m V_m}(\check{Z}(K)) \\ &= -\frac{c_{n;n}}{2} \sum [m] \hat{W}_{sl_2;V_m}(\hat{Z}(K)) \\ &= -\frac{c_{n;n}}{2} \sum [m] Q^{sl_2;V_m}(K) \end{aligned} \tag{2.16}$$

where we obtain the second and third equality by Corollary 1.5 and Theorem 1.2, respectively; recall that $\check{Z}(K)$ is defined to be $\hat{Z}(K) \# v$.

The maps j_n and ι_n are related by the equivalence between a dashed loop and $-2n$. Here we have

$$W_{sl_2}(\text{a dashed loop}) = 3 \underset{(r)}{=} -2n.$$

Hence we obtain[8]

$$h^n \cdot \hat{W}_{sl_2}(\iota_n(\check{Z}(K))) = -\frac{c_{n;n}}{2} \sum [m] Q^{sl_2;V_m}(K).$$

---

[5] The degree is given exactly in the proof.

[6] Note that we use the notation $=_{(r)}$ in (20) in a loose sense. To be precise, it is the congruent relation generated by the congruence modulo r in the coefficients and the congruence modulo "V_r". We deal with the congruence precisely in the proof.

[7] To be precise, we need the integrality of the coefficient of $\check{Z}(K)$ to use the formula (20). We avoid the difficulty of the integrality by reducing $\check{Z}(K)$ to quantum invariants in the proof.

[8] To be precise, we need the integrality of the coefficient of $\check{Z}(K)$ as in the previous footnote §.

By the above formula and definitions of $\hat{\Omega}(M)$ and $\tau_r^{SO(3)}(M)$, we have

$$|H_1(M;\mathbb{Z})|^n \hat{W}_{sl_2}(\hat{\Omega}(M)) = \hat{W}_{sl_2}\left(\frac{\iota_n(\check{Z}(L))}{\iota_n(\check{Z}(U_+))}\right) \underset{(r)}{=} \frac{\sum [m] Q^{sl_2;V_m}(K)}{\sum [m] Q^{sl_2;V_m}(U_+)} = \tau_r^{SO(3)}(M). \tag{2.17}$$

On the other hand, by definition of $\tau^{SO(3)}(M)$ we have

$$|H_1(M;\mathbb{Z})|^{n+1} \tau^{SO(3)}(M) \underset{(r)}{=} |H_1(M;\mathbb{Z})|^{n+1} \left(\frac{|H_1(M;\mathbb{Z})|}{r}\right) \tau_r^{SO(3)}(M) \underset{(r)}{=} \tau_r^{SO(3)}(M), \tag{2.18}$$

where the second equality is derived from the fact that $f^{n+1}(f/r)$ is congruent to 1 modulo r for any integer f not divisible by r.

Comparing (2.17) and (2.18), both of which hold for infinitely many r, we obtain the required formula. $\square$

We show the precise proof of Theorem 2.7 below along the above sketch. A main difficulty which is omitted in the above sketch is on the integrality of the coefficients of formulas in the proof. In the sketch we use some congruence equations modulo r, but, for example, the congruence equation in (2.16) would not hold if the coefficients of $\check{Z}(K)$ had denominators divisible by r. To avoid the difficulty, we reduce the Kontsevich invariant to quantum invariants before using the congruence equations in the proof.

To do it, we prepare the following notation. We define the invariant $\check{Q}$ by

$$\check{Q}^{sl_2;V_{m_1},\dots,V_{m_l}}(L) = \left(\prod_i \frac{[m_i]}{m_i}\right) Q^{sl_2;V_{m_1},\dots,V_{m_l}}(L) \in \mathbb{Q}[q^{1/2}, q^{-1/2}] \tag{2.19}$$

for a framed link L and irreducible representations $V_{m_1}, \dots, V_{m_l}$ of sl_2. Further, we linearly extend the invariant to $\check{Q}^{sl_2;R_1,\dots,R_l}(L)$ for $R_i \in R(sl_2)$ by

$$\check{Q}^{sl_2;\dots,\lambda R+\lambda' R',\dots}(L) = \lambda \check{Q}^{sl_2;\dots,R,\dots}(L) + \lambda' \check{Q}^{sl_2;\dots,R',\dots}(L),$$

for $R, R' \in R(sl_2)$ and scalars λ, λ'. By Theorem 1.2 and by the above linearity, we have

$$\hat{W}_{sl_2;R_1,R_2,\dots,R_l}(\check{Z}(L)) = \check{Q}^{sl_2;R_1,R_2,\dots,R_l}(L)|_{q=e^h}. \tag{2.20}$$

Proof of Theorem 2.7. We will show the required formula in the case that M is obtained from S^3 by Dehn surgery along an algebraically split framed link L; we reduce general cases to the above case as follows. For a rational homology 3-sphere M, there exist lens spaces $L(k_1, 1), \dots, L(k_\mu, 1)$ such that $M \# L(k_1, 1) \# \cdots \# L(k_\mu, 1)$ is obtained by Dehn surgery along an algebraically split framed link, see [29]. Further, we have $\tau^{SO(3)}(L(k_i, 1)) \neq 0$, as in [29]. Since both sides of the required formula are multiplicative with respect to connected sum of 3-manifolds, we obtain general case.

Suppose that M is obtained by Dehn surgery along an algebraically split framed link with l components. By Proposition 2.3, we have

$$j_n(\check{Z}(L)) = \left(\frac{1}{n!}\,\varepsilon \circ p(\alpha)^n\right)^{\otimes l}(\check{Z}(L))$$
$$= \sum_{n_1,\ldots,n_l \geqslant n} c_{n_1;n} \cdots c_{n_l;n}((\varepsilon \circ \alpha^{n_1}) \otimes \cdots \otimes (\varepsilon \circ \alpha^{n_l}))(\check{Z}(L))$$

where we put $p(\alpha)^n/n! = \sum_{i=n}^{\infty} c_{i;n}\alpha^i$. Note that $c_{i;n} \in \mathbb{Z}[1/2, 1/3, \ldots, 1/(2i+1)]$ by definition of $p(\alpha)$.

We restrict our attention to terms of at most degree N, fixing a positive integer N. Then, since chord diagrams of degree $\leqslant N$ vanish by $\alpha^{n_1} \otimes \cdots \otimes \alpha^{n_l}$ for $n_1 + \cdots + n_l > N$ by Lemma 2.1, we can reduce the right hand side of the above formula to a finite sum as follows:

$$j_n(\check{Z}(L)) = \sum_{\substack{n_1,\ldots,n_l \geqslant n \\ n_1;\cdots+n_l \leqslant N}} c_{n_1;n} \cdots c_{n_l;n}((\varepsilon \circ \alpha^{n_1}) \otimes \cdots \otimes (\varepsilon \circ \alpha^{n_l}))(\check{Z}(L))$$
$$+ (\text{terms of degree} > N).$$

We replace α^{n_i} with α^{n_i}, applying Lemma 2.6 to this formula, to obtain

$$h^{nl} \cdot \hat{W}_{sl_2}(j_n(\check{Z}(L))) = \sum_{\substack{n_1,\ldots,n_l \geqslant n \\ n_1;\cdots+n_l \leqslant N}} c_{n_1;n} \cdots c_{n_l;n} \hat{W}_{sl_2;a^{n_1},\ldots,a^{n_l}}(\check{Z}(L)) + O(h^{N+1}; \mathbb{Q}[[h]])$$
$$= \sum_{\substack{n_1,\ldots,n_l \geqslant n \\ n_1;\cdots+n_l \leqslant N}} c_{n_1;n} \cdots c_{n_l;n}\, \check{Q}^{sl_2;a^{n_1},\ldots,a^{n_l}}(L)|_{q=e^h} + O(h^{N+1}; \mathbb{Q}[[h]]) \qquad (2.21)$$

where the first factor h^{nl} is derived from the fact that j_n decreases the degree of chord diagrams by nl, and the second equality is obtained by (2.20). By substituting an indeterminate s, we have

$$(h^{nl} \cdot \hat{W}_{sl_2}(j_n(\check{Z}(L))))\Big|_{h=\log(s+1)}^{(\leqslant N)} = \left(\sum_{\substack{n_1,\ldots,n_l \geqslant n \\ n_1;\cdots+n_l \leqslant N}} c_{n_1;n} \cdots c_{n_l;n}\, \check{Q}^{sl_2;a^{n_1},\ldots,a^{n_l}}(L)\right)\Bigg|_{q=s+1}^{(\leqslant N)}$$

where the power $(\leqslant N)$ implies the operation of cutting off the terms of degree (i.e., power of s in this case) $> N$, and the subscript on the right-hand side implies the substitution of $(s+1)^{m/2} = \sum_{k=0}^{\infty} \binom{m/2}{k} s^k \in \mathbb{Z}[1/2][[s]]$ into $q^{m/2}$, where $\binom{m/2}{k}$ denotes the extended binomial coefficient defined by

$$\binom{m/2}{k} = \frac{m/2(m/2-1)\cdots(m/2-k+1)}{k!} \in \mathbb{Z}\left[\frac{1}{2}\right], \qquad (2.22)$$

for any integer m.

We fix a prime number $r \geqslant 5$, and put

$$n = \frac{r-3}{2}$$

$$N = nl + \frac{r-3}{2} = (l+1)\,\frac{r-3}{2}$$

in the following of this proof. Let ζ be $\exp(2\pi\sqrt{-1}/r)$. Unless $n_1 = n_2 = \cdots = n_l = n$, the invariant $\check{Q}^{sl_2;a^{n_1},\ldots,a^{n_l}}(L)$ vanishes by Lemma 2.8 below. Hence we have

$$\left(h^{nl}\cdot \hat{W}_{sl_2}(j_n(\check{Z}(L)))\Big|_{h=\log(s+1)}^{(\leqslant N)}\right)\Bigg|_{s=\zeta-1}$$

$$=\left(-\frac{c_{n;n}}{2}\right)^l\left(\check{Q}^{sl_2;\Sigma mV_m,\ldots,\Sigma mV_m}(L)\Big|_{q=s+1}^{(\leqslant N)}\right)\Bigg|_{s=\zeta-1} + O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta])$$

$$=\left(-\frac{c_{n;n}}{2}\right)^l\check{Q}^{sl_2;\Sigma mV_m,\ldots,\Sigma mV_m}(L)\Big|_{q=\zeta} + O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta])$$

$$=\left(-\frac{c_{n;n}}{2}\right)^l\sum_{m_1,\ldots,m_l}[m_1]\cdots[m_l]Q^{sl_2;V_{m_1},\ldots,V_{m_l}}(L)\Big|_{q=\zeta} + O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta])$$

where we obtain the second equality by applying Lemma 2.9 below to $\check{Q}^{sl_2;\Sigma mV_m,\ldots,\Sigma mV_m}(L)\in\mathbb{Z}_{(r)}[q^{1/2},q^{-1/2}]$, and obtain the third equality by definition of $\check{Q}$, Here, the sum in the formulas runs over all odd m satisfying $1\leqslant m\leqslant r-2$ as in the statement of Lemma 2.5. We replace $j_n(\check{Z}(L))$ with $\iota_n(\check{Z}(L))$ by Lemma B.1 proved in Appendix B, noting that r is divisible by $(\zeta-1)^{r-1}$ in $\mathbb{Z}[\zeta]$. Then, we have

$$\left(h^{nl}\cdot \hat{W}_{sl_2}(\iota_n(\check{Z}(L)))\Big|_{h=\log(s+1)}^{(\leqslant N)}\right)\Bigg|_{s=\zeta-1}$$

$$=\left(-\frac{c_{n;n}}{2}\right)^l\sum[m_1]\cdots[m_l]Q^{sl_2;V_{m_1},\ldots,V_{m_l}}(L)|_{q=\zeta} + O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta]).$$

Since the left-hand side (before substituted $s=\zeta-1$) is divisible by s^{nl}, we have

$$(\zeta-1)^{-nl}\left(h^{nl}\cdot \hat{W}_{sl_2}(\iota_n(\check{Z}(L)))\Big|_{h=\log(s+1)}^{(\leqslant N)}\right)\Bigg|_{s=\zeta-1}$$

$$=(\zeta-1)^{-nl}\left(-\frac{c_{n;n}}{2}\right)^l\sum[m_1]\cdots[m_l]Q^{sl_2;V_{m_1},\ldots,V_{m_l}}(L)\Big|_{q=\zeta} + O((\zeta-1)^{(r-1)/2};\mathbb{Z}_{(r)}[\zeta]) \quad (2.23)$$

where both sides of the formula belong to $\mathbb{Z}_{(r)}[\zeta]$.

In particular, putting L to be the trivial knot $U_\pm$ with ± 1 framing, we have

$$(\zeta-1)^{-n}\left(h^{n}\cdot \hat{W}_{sl_2}(\iota_n(\check{Z}(U_\pm)))\Big|_{h=\log(s+1)}^{(\leqslant n+(r-3)/2)}\right)\Bigg|_{s=\zeta-1}$$

$$=(\zeta-1)^{-n}\left(-\frac{c_{n;n}}{2}\right)\sum[m]Q^{sl_2;V_m}(U_\pm)|_{q=\zeta} + O((\zeta-1)^{(r-1)/2};\mathbb{Z}_{(r)}[\zeta]).$$

Recall that $\iota_n(\check{Z}(U_\pm))$ is invertible in $\mathscr{A}(\phi)$, see [25], and $(\zeta-1)^{-n}\sum[m]Q^{sl_2;V_m}|_{q=\zeta}$ is invertible in $\mathbb{Z}[\zeta]$, see [27]. Further, $c_{n;n}=1/(4^n\cdot n!)$ is invertible in $\mathbb{Z}_{(r)}$. Hence we have

$$(\zeta-1)^n\left(h^{-n}\cdot \hat{W}_{sl_2}(\iota_n(\check{Z}(U_\pm)))^{-1}1\Bigg|_{h=\log(s+1)}^{(\leqslant n+(r-3)/2)}\right)\Bigg|_{s=\zeta-1}$$
$$=(\zeta-1)^n\left(-\frac{c_{n;n}}{2}\right)^{-1}\left(\sum[m]Q^{sl_2;V_m}(U_\pm)\right)^{-1}\Bigg|_{q=\zeta}+O((\zeta-1)^{(r-1)/2};\mathbb{Z}_{(r)}[\zeta]). \tag{2.24}$$

By definition of $\hat{\Omega}(M)$, see [25], we have

$$|H_1(M;\mathbb{Z})|^n\hat{W}_{sl_2}(\hat{\Omega}(M))$$
$$=\hat{W}_{sl_2}(\iota_n(\check{Z}(U_+))^{-\sigma_+}\iota_n(\check{Z}(U_-))^{-\sigma_-}\iota_n(\check{Z}(L)))+O(h^{n+1};\mathbb{Q}[[h]])$$
$$=(h^{-n}\cdot\hat{W}_{sl_2}(\iota_n(\check{Z}(U_+)))^{-1})^{\sigma_+}(h^{-n}\cdot\hat{W}_{sl_2}(\iota_n(\check{Z}(U_-)))^{-1})^{\sigma_-}h^{nl}\cdot\hat{W}_{sl_2}(\iota_n(\check{Z}(L)))$$
$$+O(h^{n+1};\mathbb{Q}[[h]])$$

By applying (2.23) and (2.24), we have

$$|H_1(M;\mathbb{Z})|^n\left(\hat{W}_{sl_2}(\hat{\Omega}(M))\Bigg|_{h=\log(s+1)}^{(\leqslant n+(r-3)/2)}\right)\Bigg|_{s=\zeta-1}$$
$$=\frac{\sum[m_1]\cdots[m_l]Q^{sl_2;V_{m_1},\ldots,V_{m_l}}(L)}{(\sum[m]Q^{sl_2;V_m}(U_+))^{-\sigma_+}(\sum[m]Q^{sl_2;V_m}(U_-))^{-\sigma_-}}\Bigg|_{q=\zeta}+O((\zeta-1)^{(r-1)/2};\mathbb{Z}_{(r)}[\zeta])$$
$$=\tau_r^{SO(3)}(M)+O((\zeta-1)^{(r-1)/2};\mathbb{Z}_{(r)}[\zeta]). \tag{2.25}$$

where we obtain the second equality by definition of $\tau_r^{SO(3)}(M)$, see [9].

On the other hand, by formula (2.12) characterizing $\tau^{SO(3)}(M)$, we have

$$|H_1(M;\mathbb{Z})|^{n+1}\left(\tau^{SO(3)}(M)\Bigg|_{h=\log(s+1)}^{(\leqslant(r-3)/2)}\right)\Bigg|_{s=\zeta-1}$$
$$=|H_1(M;\mathbb{Z})|^{n+1}\left(\frac{|H_1(M;\mathbb{Z})|}{r}\right)\tau_r^{SO(3)}(M)+O((\zeta-1)^{(r-1/2};\mathbb{Z}_{(r)}[\zeta])$$
$$=\tau_r^{SO(3)}(M)+O((\zeta-1)^{(r-1)/2};\mathbb{Z}_{(r)}[\zeta]) \tag{2.26}$$

where the second equality is derived from the formula $f^{(r-1)/2}(f/r)\equiv 1$ modulo r for any integer f not divisible by r; recall that we put $n=(r-3)/2$.

By (2.25) and (2.26), we have

$$|H_1(M;\mathbb{Z})|^n\left(\hat{W}_{sl_2}(\hat{\Omega}(M))\Bigg|_{h=\log(s+1)}^{(\leqslant(r-3)/2)}\right)\Bigg|_{s=\zeta-1}$$
$$=|H_1(M;\mathbb{Z})|^{n+1}\left(\tau^{SO(3)}(M)\Bigg|_{h=\log(s+1)}^{(\leqslant(r-3)/2)}\right)\Bigg|_{s=\zeta-1}+O((\zeta-1)^{(r-1)/2};\mathbb{Z}_{(r)}[\zeta]).$$

Since this formula holds for infinitely many r, we have

$$|H_1(M;\mathbb{Z})|^n \hat{W}_{sl_2}(\hat{\Omega}(M)) = |H_1(M;\mathbb{Z})|^{n+1}\tau^{SO(3)}(M)$$

as power series in h, completing the proof. $\square$

Lemma 2.8. *Let L be an algebraically split framed link with l components, and let r be an odd prime. We put* $n=(r-3)/2$, $N=nl+(r-3)/2$ *and* $\zeta=\exp(2n\sqrt{-1}/r)$. *Suppose* $n \leqslant n_1, n_2, \ldots, n_l < r$.

1. *If some of* n_i*'s are greater than n, then we have*

$$\left(\check{Q}^{sl_2;a^{n_1},a^{n_2},\ldots,a^{n_l}}(L)\Big|_{q=s+1}^{(\leqslant N)}\right)\Bigg|_{s=\zeta-1} \in O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta]).$$

2. *If all of* n_i*'s are equal to n, then we have*

$$\left(\check{Q}^{sl_2;a^{n_1},a^{n_2},\ldots,a^{n_l}}(L)\Big|_{q=s+1}^{(\leqslant N)}\right)\Bigg|_{s=\zeta-1}$$
$$=\left(-\frac{1}{2}\right)^l\left(\check{Q}^{sl_2;\Sigma mV_m,\Sigma mV_m,\ldots,\Sigma mV_m}(L)\Big|_{q=s+1}^{(\leqslant N)}\right)\Bigg|_{s=\zeta-1} + O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta]).$$

We show the proof of this lemma in Appendix A.

Lemma 2.9. *For any* $Q \in \mathbb{Z}_{(r)}[q^{1/2}, q^{-1/2}]$ *and any integer N, we have*

$$\left(Q\Big|_{q=s+1}^{(\leqslant N)}\right)\Bigg|_{s=\zeta-1} = Q|_{q=\zeta} + O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta])$$

where the subscript on the right-hand side implies the substitution of $\zeta^{m(r+1)/2}\in\mathbb{Z}_{(r)}[\zeta]$ *into* $q^{m/2}$.

Proof. Since both sides of the required formula are additive and multiplicative, it is sufficient to show the formula for $Q=q^{1/2}, q^{-1/2}$.

If $Q=q^{1/2}$, we have

$$q^{1/2}\Big|_{q=s+1}^{(\leqslant N)} = ((1+s)^{1/2})^{(\leqslant N)}$$
$$= 1+\binom{1/2}{1}s+\binom{1/2}{2}s^2+\cdots+\binom{1/2}{N}s^N$$

where the binomial coefficients are the extended ones as defined in (2.22). Putting this formula to be $f(s)$, we have

$$f(s)^2-(s+1)=s^{N+1}g(s)$$

for some $g(s)\in\mathbb{Z}_{(r)}[s]$. By substituting $s=\zeta-1$, we have

$$(f(\zeta-1)+\zeta^{(r+1)/2})(f(\zeta-1)-\zeta^{(r+1)/2})=(\zeta-1)^{N+1}g(\zeta-1).$$

Since the first factor is equal to $2+(\zeta-1)+\cdots$, it is invertible in $\mathbb{Z}_{(r)}[s]$. Hence, we have

$$f(\zeta-1)=\zeta^{(r+1)/2}+O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta]), \tag{2.27}$$

completing this case.

If $Q=q^{-1/2}$, we put

$$f'(s)=q^{-1/2}\Big|_{q=s+1}^{(\leqslant N)}=1+\binom{-1/2}{1}s+\binom{-1/2}{2}s^2+\cdots+\binom{-1/2}{N}s^N.$$

Then, we have

$$f(s)f'(s)=1+s^{N+1}g'(s)$$

for some $g'(s)\in\mathbb{Z}_{(r)}[s]$. By substituting $s=\zeta-1$, we have

$$f(\zeta-1)f'(\zeta-1)=1+(\zeta-1)^{N+1}g'(\zeta-1).$$

By formula (32), we have

$$f'(\zeta-1)=\zeta^{-(r+1)/2}+O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta])$$

completing this case. $\square$

3. The invariants λ_n's are finite type

We review the definition and basic properties of finite type invariants of integral homology 3-spheres. As a corollary of Theorem 2.7, we show that the coefficients of the perturbative $SO(3)$ invariant are finite type.

3.1. Finite type invariants of integral homology 3-spheres

Let M be an integral homology 3-sphere, and L a framed link in M. Recall that we call L algebraically split if the linking number of any two components of L is zero. Further we call L *unit-framed* if the framing of any component of L is ± 1.

Let $\mathscr{M}$ be the vector space over $\mathbb{C}$ spanned by the integral homology 3-spheres. We put

$$(M,L)=\sum_{L'\subset L}(-1)^{\# L'}M_{L'}\in\mathscr{M}$$

where the sum runs over all sublinks of L, $\# L'$ denotes the number of components of L' and $M_{L'}$ denotes the integral homology 3-sphere obtained from M by Dehn surgery along L'. We define $\mathscr{M}_d$ to be the vector subspace of $\mathscr{M}$ spanned by any (M,L) such that L is any algebraically split unit-framed d-component link in M. Then we have a filtration of $\mathscr{M}$ as $\mathscr{M}=\mathscr{M}_0\supset\mathscr{M}_1\supset\mathscr{M}_2\supset\cdots$.

As in [30] we define a linear map $v:\mathscr{M}\to\mathbb{C}$ to be *finite type of order d* if v vanishes in $\mathscr{M}_{d+1}$. By the definition the set of finite type invariant of order d is identified with the dual space of $\mathscr{M}/\mathscr{M}_{d+1}$.

Theorem 3.1 (Garoufalidis and Ohtsuki [5]). *Let d be any non-negative integer.*

1. *We have $\mathscr{M}_{3d+1} = \mathscr{M}_{3d+2} = \mathscr{M}_{3d+3}$.*
2. *There exists a surjective linear map*

$$\varphi : \mathscr{A}(\phi)^{(d)} \to \mathscr{M}_{3d}/\mathscr{M}_{3d+1}$$

where $\mathscr{A}(\phi)^{(d)}$ denotes the vector subspace of $\mathscr{A}(\phi)$ spanned by chord diagrams of degree d.

Let v be a finite type invariant of order $3d$. Then v induces a map $\mathscr{M}_{3d}/\mathscr{M}_{3d+1} \to \mathbb{C}$. The composition of φ in Theorem 3.1(2) and the above map is called the *weight system* of v.

The following theorem implies the universality of Ω among the finite type invariants of integral homology 3-spheres.

Theorem 3.2 (Le [16]). *Let d be any non-negative integer.*

1. *We have $\Omega(\mathscr{M}_{3d}) \subset \mathscr{A}(\phi)^{(\geqslant d)}$, where $\mathscr{A}(\phi)^{(\geqslant d)}$ denotes the vector subspace of $\mathscr{A}(\phi)$ spanned by chord diagrams of degree $\geqslant d$.*
2. *By* (1) *the map Ω induces the following map*:

$$[\Omega] : \mathscr{M}_{3d}/\mathscr{M}_{3d+1} \to \mathscr{A}(\phi)^{(\geqslant d)}/\mathscr{A}(\phi)^{(\geqslant d+1)} \cong \mathscr{A}(\phi)^{(d)}.$$

This map is equal to the inverse map of the map φ in Theorem 3.1 (2).

3.2. The invariants λ_n's are finite type

The following theorem was obtained in [15] by comparing definitions of the perturbative $SO(3)$ invariant and the universal perturbative invariant directly through W_{sl_2}. Here we show another proof of the theorem using Theorem 2.7.

Theorem 3.3 (Kricker and Spence [15]). *Let d be any non-negative integer, and λ_d the coefficient of the perturbative $SO(3)$ invariant as in* (2.11).

1. *The invariant λ_d is finite type of order $3d$.*
2. *Moreover, its weight system is equal to W_{sl_2}.*

Proof. We define $\hat{\lambda}_d$ by $\tau^{SO(3)}(M) = \sum_{d=0}^{\infty} \leqslant \hat{\lambda}_d(M)h^d$. We compare $\hat{\lambda}_d$ and λ_d as follows. Put $s = e^h - 1$. Then by (2.11) we have

$$\tau^{SO(3)}(M) = \sum_{d=0}^{\infty} \lambda_d s^d.$$

On the other hand, we have

$$\begin{aligned} q^{SO(3)}(M) &= \sum_{d=0}^{\infty} \hat{\lambda}_d(\log(s+1))^d \\ &= \sum_{d=0}^{\infty} \hat{\lambda}_d\left(s - \frac{s^2}{2} + \frac{s^3}{3} - \cdots\right)^d \end{aligned}$$

by (2.13). Hence λ_d is equal to the sum of $\hat{\lambda}_d$ and a linear sum of $\hat{\lambda}_0, \hat{\lambda}_1, \ldots, \hat{\lambda}_{d-1}$. Therefore it is sufficient to show the theorem for $\hat{\lambda}_d$ instead of λ_d.

By Theorem 2.7 we have the following commutative diagram, noting that Ω and $\hat{\Omega}$ are equal for integral homology 3-spheres.

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "\Omega"'] \arrow[dr, "\tau^{SO(3)}(\cdot)"] & \\ \mathcal{A}(\phi) \arrow[rr, "\hat{W}_{sl_2}"'] & & \mathbb{C}[[h]] \end{tikzcd} \tag{3.1}$$

By restricting it to $\mathcal{M}_{3d}$, we obtain the following commutative diagram,

$$\begin{tikzcd} & \mathcal{M}_{3d} \arrow[dl, "\Omega"'] \arrow[dr, "\tau^{SO(3)}(\cdot)"] & \\ \mathcal{A}(\phi)^{(\geqslant d)} \arrow[rr, "\hat{W}_{sl_2}"'] & & h^d \cdot \mathbb{C}[[h]], \end{tikzcd} \tag{3.2}$$

where the image of Ω is included in $\mathcal{A}(\phi)^{(\geqslant d)}$ by Theorem 3.2(1), and the image of $\tau^{SO(3)}$ is included in $h^d \cdot \mathbb{C}[[h]]$ by the commutativity of the diagram. Replacing d by $d+1$, we obtain

$$\begin{tikzcd} & \mathcal{M}_{3d+1} \arrow[dl, "\Omega"'] \arrow[dr, "\tau^{SO(3)}(\cdot)"] & \\ \mathcal{A}(\phi)^{(\geqslant d+1)} \arrow[rr, "\hat{W}_{sl_2}"'] & & h^{d+1} \cdot \mathbb{C}[[h]], \end{tikzcd} \tag{3.3}$$

noting that $\mathcal{M}_{3d+1} = \mathcal{M}_{3d+3}$ by Theorem 3.1(1). Since $\hat{\lambda}_d$ is the coefficient of h^d in $\tau^{SO(3)}$, it vanishes in $\mathcal{M}_{3d+1}$. This implies that $\hat{\lambda}_d$ is finite type of order $3d$, completing the proof of (1).

By dividing the diagram (3.2) by (3.5), we obtain the following commutative diagram

$$\begin{tikzcd} & \mathcal{M}_{3d}/\mathcal{M}_{3d+1} \arrow[dl, shift left, "{[\Omega]}"] \arrow[dr, "\hat{\lambda}_d"] & \\ \mathcal{A}(\phi)^{(d)} \arrow[ur, shift left, "\varphi"] \arrow[rr, "W_{sl_2}"'] & & \mathbb{C} \end{tikzcd} \tag{3.4}$$

where the right map becomes $\hat{\lambda}_d$ because the map is equal to

$$\mathcal{M}_{3d}/\mathcal{M}_{3d+1} \xrightarrow{\tau^{SO(3)}} \frac{h^d \cdot \mathbb{C}[[h]]}{h^{d+1} \cdot \mathbb{C}[[h]]} \cong \mathbb{C} \tag{3.5}$$

where the composed map is equal to $\hat{\lambda}_d$. Further by Theorem 3.2(2) we have the inverse map φ of $[\Omega]$.

Taking the dual of the above diagram (3.4), we have the following commutative diagram:

$$\begin{tikzcd}
 & \dfrac{\{\text{the finite type invariants of degree } 3d\}}{\{\text{the finite type invariants of degree } 3d-1\}} \arrow[dl, "\varphi^*"'] & \\
\mathcal{A}(\phi)^{(d)})^* & & \mathbb{C} \arrow[ll, "W_{sl_2}{}^*"] \arrow[ul, "\hat{\lambda}_d^*"']
\end{tikzcd}$$

We compare two images of $1 \in \mathbb{C}$ in $(\mathscr{A}(\phi)^{(d)})^*$. On one hand, $\hat{\lambda}_d^*(1)$ is the Vassiliev invariant induced by $\hat{\lambda}_d$. Hence the image of $\hat{\lambda}_d^*(1)$ in $(\mathscr{A}(\phi)^{(d)})^*$ is equal to the weight system of the finite type invariant. On the other hand, the image of $1 \in \mathbb{C}$ by $W_{sl_2}^*$ is the map W_{sl_2} itself. The required equality of (2) is the equality of these two images of 1; it is derived from the commutativity of the above diagram. □

Appendix A. Proof of Lemma 2.8

In this appendix, we show the proof of Lemma 2.8.

Proof. By Lemma A.2 below, we replace each a_{n_i} with $-(1/2)\sum mV_m$ or one of linear sums of representations given in (A6).

Step 1. If some of a_{n_i}'s, say k terms, are replaced by V_r, then, by definition of $\check{Q}$,

$$\check{Q}^{sl_2;\,-(1/2)\Sigma mV_m,\,\ldots,\,rV_m,\,\ldots,\,V_{r-m}+V_{r+m},\,\ldots,\,V_r,\,\cdots}(L) \tag{A1}$$

is equal to $([r]/r)^k Q'$ with some $Q' \in \mathbb{Z}_{(r)}[q^{1/2}, q^{-1/2}]$, where Q' is a linear sum of $Q^{sl_2;\,\ldots,V_r,\cdots}(L)$'s. We put $P(s)$ to be the polynomial $Q'|_{q=s+1}^{(\leqslant N)}$. By Lemma 2.9, we have $P(\zeta-1) \in O((\zeta-1)^{N+1}; \mathbb{Z}_{(r)}[\zeta])$, since $Q'|_{q=\zeta}=0$; we show this formula as follows. By [9, Lemma 3.29] $Q^{sl_2;\,\ldots,V_r,\cdots}(L)$ is divisible by $[r]$. Since $[r]|_{q=\zeta}=0$, we have $Q'|_{q=\zeta}=0$.

Further, with power series $f(s)$ and $g(s)$ given in Lemma A.1 below, we have

$$\left(\frac{[r]}{r}\right)^k Q' \Bigg|_{q=s+1}^{(\leqslant N)} = \left(\left(f(s) + \frac{s^{r-1}}{r}\,g(s)\right)^k P(s)\right)^{(\leqslant N)}.$$

By putting $s=\zeta-1$, we have

$$\left(\left(\frac{[r]}{r}\right)^k Q' \Bigg|_{q=s+1}^{(\leqslant N)}\right)\Bigg|_{s=\zeta-1} \in O((\zeta-1)^{N+1}; \mathbb{Z}_{(r)}[\zeta])$$

noting that $(\zeta-1)^{r-1}$ is divisible by r in $\mathbb{Z}[\zeta]$. Hence, if some of a_{n_i}'s are replaced by V_r, we can ignore the terms in (A1).

In the following of this proof, we consider only the case that each a_{n_i}'s is replaced with $-(1/2)\sum mV_m$ or the first or the second linear sum of representations in (A6).

Step 2. If some of a_{n_i}'s are replaced by $V_{r-m}+V_{r+m}$, then we have

$$Q^{sl_2;\dots,V_{r-m}+V_{r+m},\dots}(L)|_{q=\zeta}=0$$

since $V_{r-m}+V_{r+m}$ is divisible by V_r in the representation ring of sl_2; recall the argument using [9, Lemma 3.29] in Step 1. Further, by definition of $\check{Q}$, we have

$$\check{Q}^{sl_2;\dots,V_{r-m}+V_{r+m},\dots}(L)$$

$$=\frac{[r-m]}{r-m}\,Q^{sl_2;\dots,V_{r-m},\dots}(L)+\frac{[r+m]}{r+m}\,Q^{sl_2;\dots,V_{r-m},\dots}(L)$$

$$=\frac{[r+m]}{r+m}\,Q^{sl_2;\dots,V_{r-m}+V_{r+m},\dots}(L)+\left(\frac{[r-m]}{r-m}-\frac{[r+m]}{r+m}\right)Q^{sl_2;\dots,V_{r-m},\dots}(L) \tag{A2}$$

The first term on the right-hand side of (A2) vanishes when $q=\zeta$. By the argument using Lemma 2.9 in Step 1 we have

$$\left(\frac{[r+m]}{r+m}\,Q^{sl_2;\dots,V_{r-m}+V_{r+m},\dots}(L)|^{(\leqslant N)}_{q=s+1}\right)\Bigg|_{s=\zeta-1}\in O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta]). \tag{A3}$$

The second term on the right-hand side of (A2) is divisible by r, noting $[r-m]/(r-m)-[r+m]/(r+m)$ is divisible by r. Hence we have

$$\left(\left(\frac{[r-m]}{r-m}-\frac{[r+m]}{r+m}\right)Q^{sl_2;\dots,V_{r-m},\dots}(L)\Bigg|^{(\leqslant N)}_{q=s+1}\right)\Bigg|_{s=\zeta-1}\in O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta]) \tag{A4}$$

by the argument in Step 3 below. By taking the sum of (40) and (41) we have

$$\left(\check{Q}^{sl_2;\dots,V_{r-m}+V_{r+m},\dots}(L)\Bigg|^{(\leqslant N)}_{q=s+1}\right)\Bigg|_{s=\zeta-1}\in O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta])$$

completing Step 2.

In the following of this proof, we consider only the case that each a_{n_i}'s are replaced by $r\cdot V_m$ or $-(1/2)\sum mV_m$.

Step 3. If some of a_{n_i}'s, say k terms, are replaced by $r\cdot V_m$, then

$$\check{Q}^{sl_2;\Sigma mV_m,\dots,\Sigma mV_m,rV_{m_{l-k+1}},\dots,rV_{m_l}}(L) \tag{A5}$$

is equal to

$$\frac{[m_{l-k+1}]}{m_{l-k+1}}\cdots\frac{[m_l]}{m_l}\,r^k\sum_{m_1,\dots,m_{l-k}}[m_1]\cdots[m_{l-k}]\,Q^{sl_2;V_{m_1},\dots,V_{m_{l-k}}V_{m_{l-k+1}},\dots,V_{m_l}}(L).$$

By Lemma A.3 below, (A5) is divisible by $\zeta-1$ sufficiently many times. Hence, we have

$$\left(\check{Q}^{sl_2;\Sigma mV_m,\dots,\Sigma mV_m,rV_{m_{l-k+1}},\dots,rV_{m_l}}(L)\Bigg|^{(\leqslant N)}_{q=s+1}\right)\Bigg|_{q=s+1}\in O((\zeta-1)^{N+1};\mathbb{Z}_{(r)}[\zeta]),$$

completing this case.

Step 4. The remaining case is that all of a_{n_i}'s are replaced with $-(1/2)\sum mV_m$. This case gives the first term on the right-hand side of the required formula of the second part of Lemma 2.8. As for the first part of Lemma 2.8, we do not have this case. $\square$

Lemma A.1. *For any odd prime r, the following formula holds*:

$$\left.\frac{[r]}{r}\right|_{q=s+1} = f(s) + \frac{s^{r-1}}{r}\, g(s)$$

for some power series $f(s)$, $g(s) \in \mathbb{Z}[[s]]$.

Proof. By definition of $[r]$, we have

$$[r] = \frac{q^{r/2} - q^{-r/2}}{q^{1/2} - q^{-1/2}} = \frac{q^r - 1}{q - 1}\, q^{(1-r)/2}.$$

By substituting $q = s + 1$, we have

$$[r]|_{q=s+1} = \left(\binom{r}{1} + \binom{r}{2} s + \cdots + \binom{r}{r} s^{r-1}\right)(q^{(1-r)/2}|_{q=s+1}).$$

Since the binomial coefficient $\binom{r}{k}$ is divisible by r for $k < r$, we obtain the required formula. $\square$

Lemma A.2. *Let r be an odd prime.*

1. *If $(r-3)/2 < m < r$, then a^m is equal to a sum of $-(1/2)\sum mV_m$ and a linear sum of*

$$\begin{aligned} &r \cdot V_m \quad \textit{for } 0 < m < r,\ r < m < 2r \\ &V_{r-m} + V_{r+m} \quad \textit{for } 0 < m < r, \textit{ and} \\ &V_r \end{aligned} \tag{A6}$$

   with coefficients in $\mathbb{Z}[1/2]$.
2. *If $m = (r-3)/2$, then a^m is equal to a linear sum of representations in* (*A*6) *with coefficients in $\mathbb{Z}[1/2]$.*

Proof. This lemma is equivalent to Lemma 2.5 via elementary calculations of representations. $\square$

Lemma A.3.

$$\sum_{\substack{1 \leqslant m_1, \ldots, m_j < r \\ m_1, \ldots, m_j \text{ are odd}}} [m_1] \cdots [m_j] Q^{sl_2; V_{m_1}, \ldots, V_{m_j}, V_{m_{j+1}}, \ldots, V_{m_l}}(L)|_{q=\zeta}.$$

is divisible by $(\zeta - 1)^{j(r-3)/2}$ in $\mathbb{Z}[\zeta]$.

Proof. Since the formula belongs to $\mathbb{Z}[\zeta]$, it is sufficient to show the claim that it is divisible by $(\zeta - 1)^{j(r-3)/2}$ in $\mathbb{Z}_{(r)}[\zeta]$. This claim was obtained for $j = l$ in [27, 29]. For $j < l$, the claim also holds, since the formula (3.3) in [29] is also valid even if we restrict the range of the sum. $\square$

Appendix B. Difference between j_n and ι_n

The aim of this appendix is to prove the following lemma, which is proved in the end of this appendix.

Lemma B.1. *Let L be an algebraically split framed link. Let r be an odd prime, and put $n=(r-3)/2$. Then,*

$$\hat{W}_{sl_2}(j_n(Z(L))-\iota_n(Z(L)))\Big|_{h=\log(s+1)}^{(\leqslant(r-3)/2)}$$

belongs to $\mathbb{Z}_{(r)}[s]$. Further, it is divisible by r in $\mathbb{Z}_{(r)}[s]$. Recall that the right subscript implies substitution of $\log(s+1)\in\mathbb{Q}[[s]]$ into h, and the right power implies cutting of the part of degree $>(r-3)/2$.

Remark B.2. Thang Le [17] showed that $j_n(\check{Z}(L))$ is equal to a linear sum of chord diagrams, such that the prime factors of the denominator of the coefficient of each chord diagram in the sum is at most $2d+1$, where d is the degree of the chord diagram. If we use the result, we obtain Lemma B.1 immediately as follows. By definition of ι_n, we obtain $\iota_n(\check{Z}(L))$ from $j_n(\check{Z}(L))$ by replacing a dashed loop u with $-2n$; note that $W_{sl_2}(u)=3$ is congruent to $-2n$ modulo r. Since denominators of the coefficients do not have the factor r in low degrees, $j_n(\check{Z}(L))-\iota_n(\check{Z}(L))$ is divisible by r; this implies Lemma B.1.

In the following of this appendix, we give another proof of Lemma B.1 without using the above result by Thang Le.

Let z be an element in $\mathscr{A}(\sqcup^l I)$, where I denotes an interval, such that

$$z=f_1\Theta_1+f_2\Theta_2+\cdots+f_l\Theta_l+w \tag{B1}$$

where Θ_i is the chord diagram on $\sqcup^l I$ with one isolated dashed arc on the ith I, and w is a linear sum of chord diagrams, each of which consists of a connected dashed component on $\sqcup^l I$ with at least one trivalent vertex. With the algebra structure in $\mathscr{A}(\sqcup^l I)$, we have the element $e^z=1+z+z^2/2+\cdots\in\mathscr{A}(\sqcup^l I)$. Further, we denote by $\hat{e}^z$ the element in $\mathscr{A}(\sqcup^l S^1)$ obtained from e^z by closing each I.

Note that, for any algebraically split framed link L, we can put $\check{Z}(L)=\hat{e}^z$ for some z as in (B1); this is shown as follows. By cutting all components of L, we obtain a tangle T consisting of l arcs such that the closure of T is isotopic to L. Since the modified Kontsevich invariant of the closing l arcs is equal to $\Delta^{(l-1)}(l)$, we obtain $\check{Z}(L)$ as the closure of $\Delta^{(l-1)}(l)\cdot\check{Z}(T)$. Further, since $\Delta^{(l-1)}(l)\cdot\check{Z}(T)$ is group like, we have a primitive element $z\in\mathscr{A}(\sqcup^l I)$ satisfying $\Delta^{(l-1)}(l)\cdot\check{Z}(T)=e^z$. This implies $\check{Z}(L)=\hat{e}^z$. Moreover, we show that z is taken as in the formula (B1) as follows. Since z is primitive, it is equal to a linear sum of chord diagrams, each of which has a connected dashed component. Further, if the dashed component has no trivalent vertices, it is a dashed arc on $\sqcup^l I$. Furthermore, if the arc connects two different I's, the coefficient of such chord diagram implies the linking number of corresponding two components of T; it vanishes in this case, since L is algebraically split. Hence, we have such z as in formula (B1) satisfying $\check{Z}(L)=\hat{e}^z$.

The following lemma for $l=1$ was suggested by Thang Le.

Lemma B.3. *Let z be the element given in the formula (B1). Let $j_{n;k} : \mathscr{A}(\sqcup^l S^1) \to \mathscr{A}(\sqcup^{l-1} S^1)$ be the map j_n acting on the kth S^1, i.e., $j_{n;k}$ is the map replacing the kth solid circle with m dashed univalent vertices with T^n_m. Then we have*

$$j_{n;k}(\hat{e}^z) = \sum_{i=1}^{n} \frac{f_k^i}{i!} u(u-2)(u-4)\cdots(u-2i+2)\cdot j_{n-i;k}(\hat{e}^z)$$
$$+ (\textit{terms without dashed loops})$$

where u denotes the dashed loop, and we regard $j_{0;k}(\hat{e}^z)$ as $\varepsilon_k(\hat{e}^z)$. Here the map ε_k: $\mathscr{A}(\sqcup^l S^1) \to \mathscr{A}(\sqcup^{l-1} S^1)$ is the map e *acting on the kth S^1.*

Proof. For $n = 1$, the kth solid circle is replaced with T_m by the map $j_{n;1}$. Recall that T_m has dashed trivalent vertices for $m > 2$ and $T_0 = T_1 = 0$. Hence, T_m contributes the terms including a dashed loop only when $m = 2$. Therefore, the term in $\hat{e}^z$ for the contribution is $f_k \Theta_k \cdot \varepsilon_k(\hat{e}^z)$, and the term from the contribution is equal to $f_k u \cdot \varepsilon_k(\hat{e}^z)$, since we have $j_1(\Theta) = u$, where Θ denotes the chord diagram consisting of a solid circle and an isolated dashed chord. Hence, we have

$$j_{1;k}(\hat{e}^z) = f_k u \cdot \varepsilon_k(\hat{e}^z) + (\text{terms without dashed loops}) \tag{B2}$$

completing the case $n = 1$.

For $n = 2$, recall that T^2_m is 1/2 times the sum of the shuffle product of T_{m_1} and T_{m_2} for $m_1 + m_2 = m$, see [25]. The graph T_{m_1} (resp. T_{m_2}) contributes terms including dashed loops in $j_{2;k}(\hat{e}^z)$, only when $m_1 = 2$ (resp. $m_2 = 2$), as in the case $n = 1$. There are the following two cases for the contribution. The one is the case that exactly one of T_{m_1} and T_{m_2} contributes the terms including dashed loops. In this case, the terms in $j_{2;k}(\hat{e}^z)$ from the contribution is equal to $f_k u \cdot j'_1$ (where we put the remaining term in the equation (B2) to be j'_1, that is, $j'_1 = j_{1;k}(\hat{e}^z) - f_k u$), because the contribution from T_2 is equal to $f_k u$ as in the case $n = 1$ and the contribution from T_{m-2} is equal to j'_1. Note that, though there are two cases for choosing $m_1 = 2$ or $m_2 = 2$, this "two" cancels with 1/2 in the definition of T^2_m. The other is the case that both of T_{m_1} and T_{m_2} contribute the terms including dashed loops. In this case, the terms in $j_{2;k}(\hat{e}^z)$ from the contribution is $(f_k^2/2)u(u+2)\cdot\varepsilon_k(\hat{e}^z)$, since $(f_k^2/2)\Theta_k^2 \cdot \varepsilon_k(\hat{e}^z)$ is the term in $\hat{e}^z$ for the contribution and we have $j_2(\Theta^2) = u(u+2)$. Hence, we have

$$j_{2;k}(\hat{e}^z) = f_k u \cdot j'_1 + \frac{f_k^2}{2} u(u+2)\cdot \varepsilon_k(\hat{e}^z) + (\text{terms without dashed loops}).$$

With the definition of j'_1, we obtain the required formula for $n = 2$.

Similarly for general n, we have

$$j_{n;k}(\hat{e}^z) = j'_n + \sum_{i=1}^{n} \frac{f_k^i}{i!} u(u+2)(u+4)\cdots(u+2i-2)\cdot j'_{n-i} \tag{B3}$$

by induction on n, where $j'_{n'}$ denotes the sum of terms without dashed loops in $j_{n';k}(\hat{e}^z)$. To obtain the formula, we use $j_i(\Theta^i) = u(u+2)(u+4)\cdots(u+2i-2)$ and repeat the same argument as above, considering the number of $T_{m_1}, T_{m_2}, \ldots, T_{m_n}$ which contribute the terms including dashed loops. We obtain the required formula from formula (B3) by removing j'_{n-i} in order, with formula (B3) putting n to be $n - i$. $\square$

Applying the argument of the proof of the above lemma to all components of $\sqcup^l S^1$, we have

Lemma B.4. *$(j_{n_1;1} \circ j_{n_2;2} \circ \cdots \circ j_{n_l;l})(\hat{e}^z)$ is equal to a sum of terms without dashed loops and a linear sum of $(j_{n_1';1} \circ j_{n_2';2} \circ \cdots \circ j_{n_l';l})(\hat{e}^z)$ for $n_k' < n_k$ with coefficients in $\mathbb{Z}[u, f_1, f_2, \ldots, f_l, 1/2, 1/3, \ldots, 1/(\max n_l)]$.*

Proof. If $n_2 = n_3 = \cdots = n_l = 0$, then we obtain this lemma by Lemma B.3.

Suppose that this lemma holds for $n_k = n_{k+1} = \cdots = n_l = 0$. Then we obtain the case $n_{k+1} = n_{k+2} = \cdots = n_l = 0$, in the same way as the proof of Lemma B.3, since w never contributes terms including dashed loops.

By induction on k, we obtain this lemma. □

Let $o_n : \mathring{\mathscr{A}}(\sqcup^l S^1) \to \mathscr{A}(\sqcup^l S^1)$ be the map replacing a dashed loop with the scalar $-2n$. Further, let t be the map defined by $t(D) = D - o_n(D)$, regarding $\mathscr{A}(\sqcup^l S^1)$ as a subset of $\mathring{\mathscr{A}}(\sqcup^l S^1)$. Note that $\psi(j_n(D)) = j_n(D) - \iota_n(D)$ by definition of ι_n. The map ψ behaves for the disjoint union of chord diagrams as

$$\psi(xy) = xy - o_n(x)o_n(y) = \psi(x)y + o_n(x)\psi(y). \tag{B4}$$

Lemma B.5. *For each positive integer n, $j_n(\hat{e}^z) - \iota_n(\hat{e}^z)$ is equal to a product of $u + 2n$ and a linear sum of $(j_{n_1;1} \circ j_{n_2;2} \circ \cdots \circ j_{n_l;l})(\hat{e}^z)$ for $n_k < n$ with coefficients in*

$$\mathbb{Z}[u, f_1, f_2, \ldots, f_l, 1/2, 1/3, \ldots, 1/n].$$

Proof. More generally, we show that $\psi((j_{n_1;1} \circ j_{n_2;2} \circ \cdots \circ j_{n_l;l})(\hat{e}^z))$ is equal to a product of $u + 2n$ and a linear sum of $(j_{n_1';1} \circ j_{n_2';2} \circ \cdots \circ j_{n_l';l})(\hat{e}^z)$ for $n_k' < n_k$; note that the former element is equal to $j_n(\hat{e}^z) - \iota_n(\hat{e}^z)$ when $n_1 = n_2 = \cdots n_l = n$.

By the above lemma, we put

$$(j_{n_1;1} \circ j_{n_2;2} \circ \cdots \circ j_{n_l;l})(\hat{e}^z) = \sum_{n_1' < n_1,\, n_2' < n_2,\, \ldots,\, n_l' < n_l} P_{n_1', n_2', \ldots, n_l'}(u) \cdot (j_{n_1';1} \circ j_{n_2';2} \circ \cdots \circ j_{n_l';l})(\hat{e}^z)$$
$$+ \text{(terms without dashed loops)}$$

with some polynomials $P_{n_1', n_2', \ldots, n_l'}(u)$ in u. Since the terms without dashed loops vanish by ψ, we have the following formula by the property (B4) of t for a product,

$$\begin{aligned}
&\psi((j_{n_1;1} \circ j_{n_2;2} \circ \cdots \circ j_{n_l;l})(\hat{e}^z)) \\
&\quad = \sum (P_{n_1', n_2', \ldots, n_l'}(u) - P_{n_1', n_2', \ldots, n_l'}(-2n)) \cdot (j_{n_1';1} \circ j_{n_2';2} \circ \cdots \circ j_{n_l';l})(\hat{e}^z) \\
&\qquad + P_{n_1', n_2', \ldots, n_l'}(-2n) \cdot \psi((j_{n_1';1} \circ j_{n_2';2} \circ \cdots \circ j_{n_l';l})(\hat{e}^z)).
\end{aligned}$$

Since $P_{n_1', n_2', \ldots, n_l'}(u) - P_{n_1', n_2', \ldots, n_l'}(-2n)$ is divisible by $u + 2n$, we reduce the proof to induction on $n_1, n_2, \ldots, n_l$, applying the hypothesis of the induction to $\psi((j_{n_1';1} \circ j_{n_2';2} \circ \cdots \circ j_{n_l';l})(\hat{e}^z))$. □

Proof of Lemma B.1. We fix an odd prime r, and put $n=(r-3)/2$. As mentioned before, we have such an element z as given in the formula (B1) satisfying $\check{Z}(L)=\hat{e}^z$. By definition of z, each f_i in (B1) is equal to half the framing number of the ith component of L; in particular, it is half integer. By Lemma B.5, we put

$$j_n(\check{Z}(L)) - \imath_n(\check{Z}(L)) = (u+2n) \sum_{n_1,\dots,n_l<n} P_{n_1,\dots,n_l}(u)\cdot(j_{n_1;1}\circ\cdots\circ j_{n_l;l})(\check{Z}(L))$$

for some polynomials $P_{n_1,\dots,n_l}(u)\in\mathbb{Z}_{(r)}[u]$. Hence $\hat{W}_{sl_2}(j_n(\check{Z}(L))-\imath_n(\check{Z}(L)))$ is equal to a product of $\hat{W}_{sl_2}(u+2n)$, which is equal to r, and a linear sum of $\hat{W}_{sl_2}((j_{n_1;1}\circ\cdots\circ j_{n_l;l})(\check{Z}(L)))$ with coefficients in $\mathbb{Z}_{(r)}$.

Hence, it is sufficient to show

$$\hat{W}_{sl_2}((j_{n_1;1}\circ\cdots\circ j_{n_l;l})(\check{Z}(L)))\Big|_{h=\log(s+1)}^{(\leqslant(r-3)/2)} \in \mathbb{Z}_{(r)}[s]. \tag{B5}$$

Since $h=\log(s+1)=\sum_{k=1}^{\infty}(-1)^{k+1}s^k/k$, the term $h^{n_1+\cdots+n_l}|_{h=\log(s+1)}$ is divisible by $s^{n_1+\cdots+n_l}$, and further

$$s^{-(n_1+\cdots+n_l)}\left(h^{n_1+\cdots+n_l}\Big|_{h=\log(s+1)}^{(\leqslant n_1+\cdots+n_l+(r-3)/2)}\right)$$

is invertible in $\mathbb{Z}_{(r)}[[s]]$. Hence, the formula (B5) is equivalent to

$$(h^{n_1+\cdots+n_l}\cdot\hat{W}_{sl_2}((j_{n_1;1}\circ\cdots\circ j_{n_l;l})(\check{Z}(L))))\Big|_{h=\log(s+1)}^{(\leqslant n_1+\cdots+n_l+(r-3)/2)} \in\mathbb{Z}_{(r)}[s].$$

By Proposition 2.3, the above formula is equal to

$$\hat{W}_{sl_2}((((\varepsilon p(\alpha)^{n_1})\circ\cdots\circ(\varepsilon p(\alpha)^{n_l}))(\check{Z}(L)))\Big|_{h=\log(s+1)}^{(\leqslant n_1+\cdots+n_l+(r-3)/2)}$$

$$= QV0\check{3}^{sl_2;a^{n_1},\dots,a^{n_l}}(L)\Big|_{q=s+1}^{(\leqslant n_1+\cdots+n_l+(r-3)/2)}$$

where the above equality is by formula (2.20). Since $n_i<n=(r-3)/2$, each a^{n_i} is equal to a linear sum of V_m for $m<r$. Hence it is sufficient to show

$$QV0\check{3}^{sl_2;V_{m_1},\dots,V_{m_l}}(L)|_{q=s+1}\in\mathbb{Z}_{(r)}[[s]].$$

By definition of $\check{Q}$, formula (24), and by the property $Q^{sl_2;V_{m_1},\dots,V_{m_l}}(L)\in\mathbb{Z}[q^{1/2},q^{-1/2}]$, we obtain the above formula. □

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